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APPLICATION OF BRAIDING SEQUENCES IV: LINK POLYNOMIALS AND GEOMETRIC INVARIANTS

by Alexander STOIMENOW

Abstract. — We apply the concept of braiding sequences to the Conway and skein polynomial, and some geometric invariants of positive links. Using degree and coefficient growth properties of the Conway polynomial, estimates of braid index and Legendrian invariants are given. We enumerate alternating (and some other classes of) links of given genus asymptotically up to constants by braid index.

Résumé. — Nous appliquons le concept de séquences de tressage aux polynômes de skein et de Conway, mais aussi à quelques invariants géométriques des entrelacs positifs. On donne des estimations pour l'indice des tresses et pour des invariants legendriens, en utilisant le degré et des propriétés de croissance des coefficients du polynôme de Conway. Nous énumérons asymptotiquement à une constante près les entrelacs alternants (et quelques autres) de genre donné par leur indice de tresses.

1. Introduction and overview of results

Positive knots have become gradually relevant, apparently not primarily because of the combinatorial property that describes them, but because they were found related to a series of different subjects, including dynamical systems [7], algebraic curves [44, 45], and singularity theory [1, 8, 33]. Positive knots play some role also in 4-dimensional QFTs [29], and in relation to the concordance invariants of knot homology (see, e.g., [20]). Let us note that the intersection of the classes of positive and alternating knots are the special alternating knots studied extensively by Murasugi; see for example [36].

The concept of braiding sequences [69] was used originally in relation to Vassiliev (finite degree) invariants [3, 4, 70, 71]. Braiding sequences were

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later related to positive and alternating knots [47, 64, 65] by means of the fact that the set of knot diagrams on which the Seifert algorithm gives a surface of given genus decomposes into finitely many such sequences. Our subject here will be to derive further consequences of this circumstance, first, to properties of the Conway polynomial ∇ of positive links. This paper is a continuation of the previous part of the work [61], where we studied the Jones V [23], HOMFLY–PT (or skein) P [18, 31] and Kauffman F [27] polynomials, and established a finiteness property for their coefficients (see Theorem 5.1 below).

In contrast to these polynomials, the coefficients of the Conway polynomial grow unboundedly on positive links. This is the subject of Section 4 (following some background given in Section 2 and, more specifically about braiding sequences, in Section 3). In particular, Theorem 4.5 gives a lower bound on all possible coefficients of ∇ of a positive link, proportional to the crossing number of the positive diagram (see also Corollary 4.9). This qualitatively improves the positivity result of Cromwell [12], and widens the case of positive braid links studied by van Buskirk [9], as well as the Casson knot invariant in [51]. The proof uses the work in [51] for knots and certain linking number inequalities.

In Section 5, we derive various applications of the work in Section 4. The increase of Conway's coefficients of positive knots, amplified by a similar property for the degree-3 Vassiliev invariant in Proposition 5.24, opposes the finiteness result of the Jones and Kauffman polynomials, and enforces a growth of their degrees (Section 5.5). For the skein polynomial, a much more efficient estimate follows from modifying the proof for the Conway polynomial. One has then the following applications to the braid index and Legendrian knot invariants [10, 19], which are the main subject of this paper. (The definition of the invariants is briefly recalled in Section 5.3.)

Theorem 1.1. — *If L is a semihomogeneous normal link with a semihomogeneous reduced diagram D of Euler characteristic $\chi(D)=\chi(L)$ and $c(D)$ crossings, then we have for the braid index $b(L)$ of L,*

$$b(L) \geqslant \frac{c(D)+1}{2} + 2\chi(D). \tag{1.1}$$

That is, the crossing number is linearly bounded in the braid index, when the constant term depends on the Euler characteristic.

"Normal" should mean no unknot split components. The notion of semihomogeneous links was introduced in [56] as a (slight) generalization of Cromwell's homogeneous links [12]. Thus they cover alternating and positive links, among others. This provides a rather wide yet simple *lower* braid index estimate. Note that it differs only by a quantity depending on χ from Ohyama's (general) *upper* estimate [39] (see (5.20)). It also gives a crossing number bound (Corollary 5.14).

As a further application of Theorem 1.1, we have, for many links, a quite explicit version of the finiteness result of Birman and Menasco in [6], and its partial rederivation in [55]. (See Section 2.1 for more notation.)

Theorem 1.2. — *The number $\beta_{b,\chi}$ of non-split semihomogeneous links of (fixed) Euler characteristic $\chi < 0$ and (increasing) braid index b satisfies the asymptotic equivalence*

$$\beta_{b,\chi} \sim_b b^{-3\chi-1}. \tag{1.2}$$

The theorem holds in this same form also for the subclasses, incl. for alternating and positive links (see Remark 5.15).

The other main topic we treat is the Bennequin inequality for Legendrian knots. If $\mathcal{L}$ is a Legendrian embedding of a link L in the standard contact space $(\mathbb{R}^3(x,y,z),\, \mathrm{d}x + y\,\mathrm{d}z)$, then its Thurston–Bennequin invariant $tb(\mathcal{L})$ and Maslov number $\mu(\mathcal{L})$ satisfy

$$tb(\mathcal{L}) + |\mu(\mathcal{L})| \leqslant -\chi(L). \tag{1.3}$$

(This notation will be kept throughout below.)

We obtain the following simple lower estimate on the unsharpness of Bennequin's inequality for semihomogeneous links.

Theorem 1.3. — *Let L be a semihomogeneous link with a reduced semihomogeneous diagram D of $c_-(D)$ negative crossings. Then, with the above notation,*

$$tb(\mathcal{L}) + |\mu(\mathcal{L})| \leqslant -\chi(L) - \frac{3 + c_-(D)}{2} \tag{1.4}$$

for $c_-(D) > 2$ (and "3" can be replaced by "2" for $c_-(D) = 2$).

This is an, again explicit, extension and simplification of a result in [55], which in turn generalized the examples of Kanda [24] of (increasingly) unsharp Bennequin inequality. In particular, now we see that as such examples any sequence of (links with) semihomogeneous diagrams having an unbounded number of negative crossings will do.

In the special case of negative links, we obtain a decreasing purely negative estimate (which is independent of χ), as follows.

Theorem 1.4. — *Let L be a negative link, except the unknot, with a reduced negative diagram D. Then*

$$tb(\mathcal{L}) + |\mu(\mathcal{L})| \leqslant -\frac{5 + c(D)}{4}.$$

The same statement holds for $\mathcal{L}$ transverse, when omitting the $\mu(\mathcal{L})$ term.

This can again be seen as an extension of the decrease result of Kanda [24], and concretizes its generalization in [55]. Along the lines of [55], one can derive such a decreasing negative bound also for a semihomogeneous link, when one chooses between mirror images (see Proposition 5.12).

1.1. Acknowledgment

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2. Preliminaries, Notations and Conventions

2.1. Generalities

The symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the integer, natural, rational, real and complex numbers, respectively. We will also write $i = \sqrt{-1}$ for the imaginary unit, in situations where no confusion (with the usage as index) arises. For a set S, the expression $|S|$ denotes the cardinality of S. In the sequel the symbol "$\subset$" denotes a not necessarily proper inclusion.

Let us write, just for the scope (and space) of the following definition, that for two positive integer sequences (a_n) and (b_n),

$$\underline{\lambda} = \liminf_{n\to\infty} \frac{a_n}{b_n}, \quad \lambda = \lim_{n\to\infty} \frac{a_n}{b_n}, \quad \overline{\lambda} = \limsup_{n\to\infty} \frac{a_n}{b_n}.$$

Let us then say/write for two such sequences (a_n) and (b_n) that they are/satisfy

$$\begin{aligned} a_n = O_n(b_n), &\quad \text{if } \overline{\lambda} < \infty, \\ \text{asymptotically equivalent, } a_n \sim_n b_n, &\quad \text{if } 0 < \underline{\lambda} \leqslant \overline{\lambda} < \infty, \\ \text{asymptotically proportional, } a_n \simeq_n b_n, &\quad \text{if } 0 < \lambda < \infty, \\ \text{asymptotically equal, } a_n \cong_n b_n, &\quad \text{if } \lambda = 1. \end{aligned} \tag{2.1}$$

An expression of the form "$a_n \to \infty$" should abbreviate $\lim_{n\to\infty} a_n = \infty$. Analogously "$a_{n_m} \to \infty$" should mean the limit for $m \to \infty$, etc.

We need next a few notations related to polynomials, which are understood in the broader sense as Laurent polynomials (i.e., variables are allowed to occur with negative exponents). Let $[X]_{t^a} = [X]_a$ be the coefficient of t^a in a polynomial $X \in \mathbb{Z}[t^{\pm 1}]$. For $X \neq 0$, let $\mathcal{C}_X = \{a \in \mathbb{Z} : [X]_a \neq 0\}$ and

$$\min\deg X = \min \mathcal{C}_X, \quad \max\deg X = \max \mathcal{C}_X,$$
$$\text{and} \quad \operatorname{span} X = \max\deg X - \min\deg X$$

be the minimal and maximal degree and span (or breadth) of X, respectively. It makes sense to set $\min\deg 0 := \infty$ and $\max\deg 0 := -\infty$. Similarly one defines for $X \in \mathbb{Z}[x_1, \dots, x_n]$ the coefficient $[X]_A$ for some monomial A in the x_i, and $\min\deg_{x_i} X$, etc.

Definition 2.1. — *For $H, G \in \mathbb{Z}[t, t^{-1}]$ we write $H \geqslant G$ (resp. $H \leqslant G$) if $[H]_i \geqslant [G]_i$ (resp. $[H]_i \leqslant [G]_i$) for all i. We say H is* positive *if $H \geqslant 0$ and* negative *if $H \leqslant 0$, and that H is* signed *if it is positive or negative.*

*A polynomial H is n-*strictly signed *if it is signed and $[H]_i \neq 0$ when $\min\deg H \leqslant i \leqslant \max\deg H$ and $i \equiv \min\deg H \pmod n$. We call H* strictly signed *if H is 1-strictly signed, and H is* (n-)strictly positive/negative *if it is (n-)strictly signed and positive/negative.*

The absolute (value) polynomial $|H|$ *of H is given by*

$$\big[|H|\big]_{t^i} := \big|[H]_{t^i}\big|.$$

Thus, for example, the polynomial $1+t^2$ is positive and 2-strictly positive, but not strictly positive.

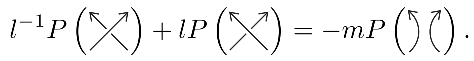
We use the abbreviations "w.l.o.g." for "without loss of generality" and "r.h.s." (resp. "l.h.s.") for "right hand-side" (resp. "left hand-side"). "w.r.t." will stand for "with respect to". Some further notations will be introduced at an appropriate place in the text.

2.2. Link polynomial invariants

As for polynomial invariants, our notation is fairly standard: Δ denotes the Alexander [2], ∇ the Conway [11], V the Jones [23], P the HOMFLY–PT (or skein) [18, 31], and F the Kauffman polynomial [27].

2.2.1. Skein (HOMFLY-PT) polynomial

The *skein (HOMFLY–PT) polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

(2.2) 

With this relation, we use the convention for P of [31], but with l and l^{-1} interchanged.

A *skein triple* D_+, D_-, D_0 is a triple of diagrams, or of their corresponding links L_+, L_-, L_0, equal except near one crossing, where they look like in (2.2) (from left to right). The replacement $L_\pm \to L_0$ is called *smoothing (out)* the crossing in $L_\pm$. The crossing in D_+ is called *positive*, the one inD_- *negative*.

Let $c(L)$, the *crossing number* of a link L, be the minimal crossing number $c(D)$ over all diagrams D of L. We write $c_+(D)$ and $c_-(D)$ for the *number of positive resp. negative crossings* of a diagram D. The sum of the signs of all crossings of D is called the *writhe* of D and will be written $w(D)$. Thus

$$c(D) = c_+(D) + c_-(D) \quad \text{and} \quad w(D) = c_+(D) - c_-(D).$$

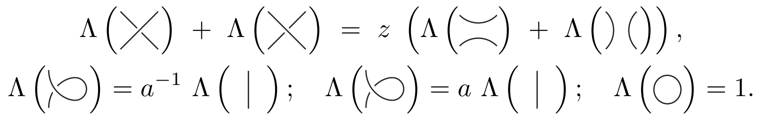
Let D be an oriented knot or link diagram. We denote by $c(D)$ the *crossing number* of D. We use $n(D) = n(L)$ to designate the *number of components* of D or its link L. We write $s(D)$ for the *number of Seifert circles* of a diagram D (the loops obtained by smoothing out all crossings of D).

2.2.2. Kauffman polynomial

The *Kauffman polynomial* [27] F is usually defined via a regular isotopy invariant $\Lambda(a,z)$ of unoriented links. For F we use the convention of [27], but with a and a^{-1} interchanged. In particular we have for a link diagram D the relation

$$F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z). \tag{2.3}$$

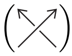
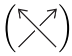
The writhe-unnormalized version Λ of F is given in our convention by the properties

(2.4) 

Thus the positive (right-hand) trefoil has $\min\deg_l P = \min\deg_a F = 2$. Note that for P and F there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

2.2.3. Jones polynomial

The *Jones polynomial* V is a Laurent polynomial in one variable t of oriented knots and links and can be defined by being 1 on the unknot and the relation

(2.5) $t^{-1}V$  $- tV$  $= (t^{1/2} - t^{-1/2})V$  .

The Jones polynomial is obtained from P and F (in our conventions) by the substitutions (with i being the complex unit; see [31] or [27, Section III])

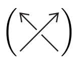
$$V(t) = P(it, i(t^{1/2} - t^{-1/2})) = F(-t^{3/4}, t^{1/4} + t^{-1/4}). \tag{2.6}$$

2.2.4. Conway–Alexander polynomial

The *Conway* and *Alexander polynomial* are equivalent, and substitutions of the skein polynomial:

$$\Delta(t) = \nabla(t^{1/2} - t^{-1/2}) = P(i, i(t^{1/2} - t^{-1/2})). \tag{2.7}$$

The skein relation for ∇ can be written

(2.8) ∇  $= \nabla$  $+ z\nabla$  .

We will often use below the coefficients

$$\nabla_j = \nabla_j(L) = [\nabla(L)]_j = [\nabla(L)]_{z^j}, \tag{2.9}$$

which we abbreviate as indicated. One can argue that $\min\deg \nabla_L(z) \geqslant n(L) - 1$, i.e., $\nabla_j(L) = 0$ whenever $j < n(L) - 1$. This in particular means that indeed $\nabla_L(z)$ is a genuine polynomial in z, and not a Laurent one, as (2.7) might suggest.

Throughout this treatise, Δ is thus normalized so that (2.7) holds. The word "normalized" refers to comparison with other definitions of the Alexander polynomial, where one often leaves an ambiguity up to units in $\mathbb{Z}[t^{\pm 1}]$.

Thus for *knots* L we will have

$$\Delta_L(1) = 1, \tag{2.10}$$

and for a general link L,

$$\Delta_L(1/t) = (-1)^{n(L)-1}\Delta(t) \tag{2.11}$$

(i.e., the sign is positive/negative for odd/even number of components). We will call Property (2.10) *unimodularity*, and (2.11) *symmetry*.

The reformulation of symmetry of Δ in terms of ∇ is that $\nabla_L(z)$ is an even/odd polynomial (i.e., has coefficients only in even/odd z-degree), when $n(L)$ is odd/even. The reformulation of unimodularity is that for knots $\nabla_0 \equiv 1$. More generally, $\nabla_{n(L)-1}(L)$ can be expressed in terms of component linking numbers (see Section 4.2 below).

2.3. Links and diagrams

Here, and in the sequel, for a knot or link K, we write $!K$ for its obverse, or mirror image. Similarly $!D$ is the mirror image of a link diagram D. By $K_1\#K_2$ we denote the connected sum of K_1 and K_2.

We say that a link diagram D is *s-almost positive* if $c_-(D) = s$. A knot or link is s-almost positive if it has an s-almost positive diagram, but no $(s-1)$-almost positive one. Hereby, for both knots and diagrams, "0-almost positive" is called shorter *positive* and "1-almost positive" is *almost positive* [52]. The procedure of turning all crossings in a diagram so that they become positive is called *positification*; a diagram obtained thus is *positivized*. *Negative* links and diagrams are defined as mirror images of positive ones.

Note: There seems some division between knot theorists as to which links are to be called positive. In [9, 32], the rather non-standard (and confusing) convention is used to call "positive knots" the knots with positive braid representations (better to be called "positive braid knots", as in [49]). The convention here follows the now established standard, used in many publications, as [12, 14, 37, 38, 40, 45, 66, 72, 73], to call positive knots the (larger) class of knots with a positive diagram.

We will use the notation $L = \sqcup_{i=1}^n L_i$ to indicate the L_i are the components of L (each L_i is thus a knot).

We call two components of a link L split-equivalent, if there is no $S^2 \subset S^3$ with $S^2 \cap L = \varnothing$ which separates these two components. This is an equivalence relation (among the components of L). A *split component* of L is a sublink made up of a split-equivalence class of components of L. A split component is *trivial* if it contains only one component of L, and this component is an unknot. A split link is a link with more than one split component. Other links are said to be non-split. We call below a link *normal* if it has no trivial (that is, unknot) split components.

By $L = L_1 \sqcup L_2$ we will designate the *split union* of L_1 and L_2. A link is split if it is a split union of two links. Note that $L_1 \sqcup L_2 = L_1 \# U_2 \# L_2$, for the 2-component unlink U_2, and thus

$$\max\deg_l P(L_1 \sqcup L_2) = \max\deg_l P(L_1) + \max\deg_l P(L_2) + 1, \tag{2.12}$$

and

$$\operatorname{span}_l P(L_1 \sqcup L_2) = \operatorname{span}_l P(L_1) + \operatorname{span}_l P(L_2) + 2. \tag{2.13}$$

A link diagram D is called *split*, or *disconnected*, if it can be non-trivially separated by a simple closed curve in the plane. Otherwise we say D is *non-split*, or *connected*. A split link is thus a link with a split diagram.

A crossing in a diagram is *reducible*, if it is transversely intersected by a simple closed curve not meeting the diagram anywhere else. A diagram is reducible if it has a reducible crossing, otherwise it is called *reduced*. To avoid confusion, unless otherwise stated, in the sequel all diagrams are assumed reduced, that is, with no nugatory crossings, and links are non-split.

A *region* of a link diagram D is a connected component of the complement of the plane curve of D. A region R of a diagram is called *Seifert circle region* (resp. non-Seifert circle, or *hole* region), if any two neighboring edges in its boundary (i.e., such sharing a crossing) are equally (resp. oppositely) oriented (between clockwise or counterclockwise) as seen from inside R. A diagram is called *special* iff all its regions are (either) Seifert circle regions or hole regions.

It is an easy combinatorial observation that for a connected diagram two of the properties alternating, positive and special imply the third. A diagram with these three properties is called *special alternating*. See, e.g., [35, 36]. A special alternating link is a link having a special alternating diagram. It can be described also (like in the introduction) as a link which is simultaneously positive and alternating. By definition such a link has a positive diagram, and an alternating diagram. That it has a diagram which enjoys simultaneously both properties was proved in [38, 54].

As in [37], every diagram decomposes under (diagrammatic, or planar) *Murasugi sum*, or $*$*-product,* into special diagrams, which may further decompose under connected sum. These components are called *Murasugi atoms* in [43]. A diagram is said to be *semihomogeneous* in [56] if all its Murasugi atoms are positive or negative. A link is semihomogeneous if it has such a diagram.

Morton [34] has shown that for every diagram

$$\max\deg_m P(D) \leqslant 1-\chi(D).$$

Murasugi–Przytycki [37] prove a multiplicativity of the maximal possible m-term in P

$$\widetilde{P}(D) = [P(D)]_{m^{1-\chi(D)}} \tag{2.14}$$

of $P(D)$ under diagrammatic Murasugi sum:

$$\widetilde{P}(D_1 * D_2) = \widetilde{P}(D_1)\cdot \widetilde{P}(D_2). \tag{2.15}$$

This result will be of crucial importance later. Note also that because of (2.7), we have a corresponding multilicativity property of $\nabla_{1-\chi}$, too (which had been known before from Murasugi's work more generally for topological Murasugi sum).

The *braid index* $b(L)$ of a link L is defined as the minimal number of strings of a braid whose closure is L; see, e.g., [34]. One main tool in estimating (and determining) the braid index is the *inequality of Morton–Williams–Franks* (MWF) [17, 34],

$$2b(L)-2 \geqslant \operatorname{span}_l P(L). \tag{2.16}$$

2.4. Genera

In the sequel we denote by $g(D)$ the *genus* of a diagram D, this being the genus of the surface coming from the Seifert algorithm applied on this diagram. More conveniently, if D is a link diagram, we use instead of $g(D)$ the notation $\chi(D)$ for the *Euler characteristic* of the Seifert surface given by the Seifert algorithm.

By $g(L)$ we will denote the genus and $\chi(L)$ the Euler characteristic of a link L, which are the minimal genus resp. maximal Euler characteristic of an orientable spanning (i.e., Seifert) surface for L. By $g_c(L)$ we denote the *canonical genus* of L, which is the minimal genus $g(D)$ of some diagram D of L. Similarly, $\chi_c(L)$, the *canonical Euler characteristic* of L, is the maximal $\chi(D)$ for all diagrams D of a link L.

Theorem 2.2 (see [12]). — *The Seifert algorithm applied on a positive diagram gives a minimal genus surface.*

Thus the genus $g(L)$ of a positive link L coincides with the genus $g(D)$ of a positive diagram D of L, given by

$$g(D) = \frac{c(D) - s(D) + 2 - n(D)}{2}, \tag{2.17}$$

with $c(D)$, $s(D)$ and $n(D) = n(L)$ being the number of crossings, Seifert circles and components of D, resp. The preceding theorem implies that for positive links $g = g_c$.

We recall two major ways of estimating genera of arbitrary knots from below. One comes from the Alexander/Conway polynomial. It is well-known that for split links $\nabla = 0$, and (as partly stated already in Section 2.2.4) for a non-split link L, the coefficient $\nabla_i(L)$ is non-zero only if $n(L) - 1 \leqslant i \leqslant 1 - \chi(L)$, and $i - n(L)$ is odd. The range of i means that (for $\nabla \neq 0$)

$$n(L) - 1 \leqslant \min\deg_z \nabla(L) \leqslant \max\deg_z \nabla(L) \leqslant 1 - \chi(L). \tag{2.18}$$

For many (non-split) links, including positive and alternating ones, the rightmost inequality is exact, i.e., an equality. (In fact, this is the way to prove Theorem 2.2, which also holds for alternating links.) Note that for knots K, with $2g(K) = 1-\chi(K)$, this inequality can be restated using (2.7) in the better-known form

$$\max\deg \Delta(K) \leqslant g(K).$$

Again, equality holds for positive (and alternating) knots K.

For positive links, the left inequality in (2.18) is also exact, as follows from the expression of $[\nabla(L)]_{n-1}$ in terms of linking numbers (see Section 4.2). This is also true for an arbitrary *knot* K, where the constant term of ∇ is 1.

Note also that (2.18) implies $2g(L) \geqslant \operatorname{span}\nabla(L)$, and equality occurs iff the leftmost and rightmost inequalities in (2.18) are both exact (as they are for positive links L).

The other way of estimating genera comes from Bennequin's Inequality [5, Theorem 3], and its subsequent improvements.

We define the *Bennequin number* $r(D)$ of a diagram D of a link L to be

$$r(D) := \frac{1}{2}\big(w(D) - s(D) + 1\big). \tag{2.19}$$

Then it is known (see [20, 45]) that

$$1 - \chi(L) \geqslant 2r(D), \tag{2.20}$$

which is called *Bennequin inequality* (and (1.3) is a "contactification" of this).

A consequence is the following. Let D be an s-almost positive diagram of a knot K. By comparison of (2.17) (with $n(D) = 1$) and (2.19), we have then

$$r(D) = g(D) - s. \tag{2.21}$$

Bennequin's inequality (2.20) becomes

$$g(D) - s = r(D) \leqslant g(K) \leqslant g_c(K) \leqslant g(D). \tag{2.22}$$

In particular, for positive diagrams ($s = 0$), all inequalities become equalities. This is one other way to Theorem 2.2.

2.5. Gauß sum invariants

We recall briefly the definition of Gauß sum invariants. They are introduced by Fiedler and Polyak–Viro, and give formulas for Vassiliev invariants [3]. We give a summary which is similar to the discussion in [51].

Definition 2.3 (see, e.g., [41]). — *A Gauß diagram of a knot diagram is an oriented circle with arrows connecting points on it mapped to a crossing and oriented from the preimage of the undercrossing (underpass) to the preimage of the overcrossing (overpass).*

We will call the two arrow ends also *hook* and *tail*.

Example 2.4. — As an example, Figure 2.1 shows the knot 6_2 in its commonly known (alternating) diagram and the corresponding Gauß diagram.

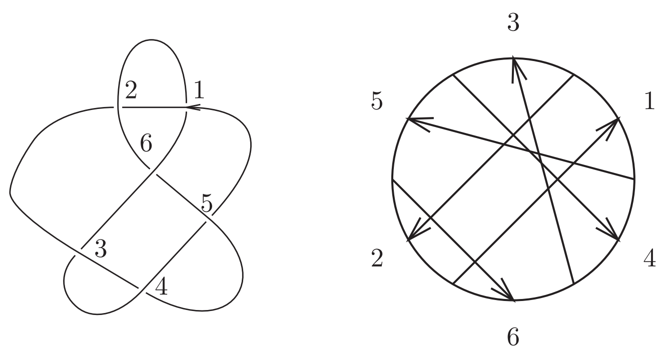
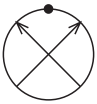
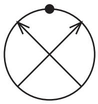
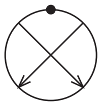


Figure 2.1. The standard diagram of the knot 6_2 *and its Gauß diagram.*

The simplest (non-trivial) Vassiliev knot invariant is the *Casson invariant* $v_2 = \nabla_2$, with the alternative expression $-6v_2 = V''(1)$. For it Polyak–Viro [41, 42] gave the simple Gauß sum formula

(2.23) $v_2 =$  $= \dfrac{1}{2}\Bigg($  $+$  $\Bigg).$

Here the point on the circle corresponds to a point on the knot diagram, to be placed arbitrarily except on a crossing. (The expression does not alter with the position of the basepoint; we will hence have, and need, the freedom to place it conveniently.)

Other formulas were given by Polyak–Viro, and also Fiedler, for the degree–3–Vassiliev invariant v_3. To make precise which variation of the degree–3–Vassiliev invariant we mean, we have

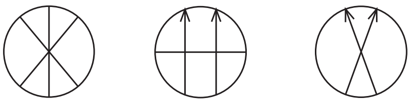
$$v_3 = -\frac{1}{12}V''(1) - \frac{1}{36}V^{(3)}(1).$$

Fiedler's formula for v_3 [15, 16] reads[(1)]

$$4v_3 = \sum_{(3,3)} w_p w_q w_r + \sum_{(4,2)0} w_p w_q w_r + \frac{1}{2}\sum_{p,\,q \text{ linked}} (w_p + w_q), \tag{2.24}$$

where the configurations are

(2.25)



$(3,3)$ $(4,2)0$ p,q linked

Here chords depict arrows which may point in both directions and w_p denotes the writhe of the crossing p. The summand for each configuration is called *weight*.

Definition 2.5. — *We call two crossings p, q in a knot diagram D* linked*, and write $p \pitchfork q$, if, passing their crossingpoints along the orientation of D, we have the cyclic order $pqpq$ (i.e., their arrows in the Gauß diagram intersect). Otherwise, if the cyclic order is $ppqq$ (and the arrows do not intersect), we call p and q* unlinked *and write $p \not\pitchfork q$.*

If two crossings p and q are linked, call *distinguished* the crossing whose over-pass (arrow head in the Gauß diagram) is followed in orientation direction (counterclockwise orientation of the circle in the Gauß diagram) by the under-pass (arrow tail) of the other crossing. (In the third diagram of (2.25) it is the arrow going from lower right to upper left.)

(1) Note the Factor 4 by which (2.24) differs from the definition in [51].

2.6. Graphs

A graph G will have for us possibly multiple edges (edges connecting the same two vertices), but usually no loop edges (edges connecting one and the same vertex). By $V(G)$ we will denote the set of vertices of G, and by $E(G)$ the set of edges of G (each multiple edge counting as a set of single edges); $v(G)$ and $e(G)$ will be the number of vertices and edges of G (thus counted), respectively. For $v \in V(G)$, we write $\mathrm{val}(v)$ for the *valence* of v in G.

An *n-cut* of G is a set $T \subset E(G)$ with $|T| = n$ such that removing (*without* their endpoints) all edges in T from G gives a disconnected graph. We say that G is *n-connected* if it has no n'-cut for $n' < n$.

For a graph, let the operation ●——● $\longrightarrow$ ●——●——● (adding a vertex of valence 2) be called *bisecting* and its inverse (removing such a vertex) *unbisecting* (of an edge). We call a graph G a *bisection* of a graph G' with no valence–2–vertices, if G is obtained from G' by a sequence of edge bisections.

We call a bisection G *reduced*, if it has no adjacent vertices of valence 2 (that is, each edge of G' is bisected at most once). Contrarily, if G is a graph, its *unbisected graph* G' is the graph with no valence–2–vertices of which G is a bisection.

Now to each link diagram D we associate its *Seifert graph* $G = G(D)$, which is a planar bipartite graph. It consists of a vertex for each Seifert circle in D and an edge for each crossing, connecting two Seifert circles. We will for convenience sometimes identify crossings/Seifert circles of D with edges/vertices of G. At least for special alternating diagrams D it is true that $G(D)$ determines D.

In [65] is was established that D is a maximal generator if and only if $G = G(D)$ is a reduced bisection of a planar 3-connected trivalent graph G'. (The confinement to knots, which were the main focus in [65], is not essential for this reasoning.)

2.7. An addendum to [58]

Definition 2.1 is needed (besides in Section 4 below) also to clarify arguments in the previous part of the work [58, Section 5], but was erronously omitted there. I apologize for this confusion.

3. Braiding sequences and genus generators

Since all work here builds on braiding sequences and generators, the preliminary account we need here is a little longer, and we devote to it a separate section. Cromwell offers in his recent book [13, Section 5.3] an introductory exposition on the subject.

3.1. Generators and generating diagrams

We start by recalling the classical flype.

Definition 3.1. — *A* flype *is a move on a diagram shown in Figure 3.1. We say that a crossing p* admits *a* flype *if it can be represented as the distinguished crossing in the diagrams in the figure, and both tangles have at least one crossing.*

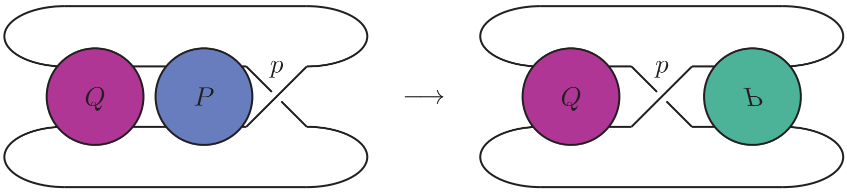
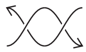


Figure 3.1. A flype near the crossing p

Note that when two diagrams differ by a flype, there is a natural bijection between either's crossings: p changes position, and the crossings in P match on either side, as do those in Q.

A *reverse clasp* is, up to crossing changes, a tangle like , and a *parallel clasp* . We call a clasp *trivial* if both its crossings have opposite sign. Such a clasp can be eliminated by a Reidemeister II move. We call the switch of a crossing in a non-trivial clasp, followed by the Reidemeister II move, *resolving* the clasp.

Now let us recall, from [47, 57], some basic facts concerning knot generators of given genus. We start by defining $\sim$ and $\stackrel{\sim}{*}$-equivalence of crossings. There are two alternative forms of these definitions, and we choose here the one that closely leans on the terminology of Gauß diagrams (see Section 2.5). We will also set up some notations and conventions used below. The situation for links is discussed only briefly here, and in much more detail in [62].

Definition 3.2. — *Let D be a* knot *diagram, and p and q be crossings.*

(i) *We call p and q twist equivalent, if (with the notation in Definition 2.5) for every $r \neq p, q$ we have $r \cap p$ if and only if $r \cap q$.*

(ii) *We call p and q* $\sim$-equivalent *and write $p \sim q$ if p and q are twist equivalent and $p \not\cap q$.*

(iii) *Similarly p and q are called* $\stackrel{\sim}{*}$-equivalent*, $p \stackrel{\sim}{*} q$, if p and q are twist equivalent and $p \cap q$.*

A minor argument will convince one that these are indeed equivalence relations. There is an alternative way of formulating them (which generalizes them to link diagrams): $p \stackrel{\sim}{*} q$ if and only if p and q can be made to form a parallel clasp after flypes. Similarly $p \sim q$ for reverse clasp.

Definition 3.3. — *A $\sim$-equivalence class consisting of one crossing is called* trivial*, a class of more than one crossing* non-trivial*. A $\sim$-equivalence class is* reduced *if it has at most two crossings; otherwise it is* non-reduced*. A $\sim$-equivalence class is* even *or* odd *depending on the parity of its crossings, and* positive *or* negative *depending on their (skein) sign (if the same for all its crossings). For the diagram D, let $t(D)$ be the number of its $\sim$-equivalence classes. A diagram is called* generating *if all its $\sim$-equivalence classes are reduced.*

The use of generating diagrams will be clarified in Section 3.3. We restate here the following result from [62], which we will need.

Theorem 3.4 ([62]). — *In a connected link diagram D of canonical Euler characteristic $\chi(D) \leqslant 0$ there are at most*

$$t(D) \leqslant d_{\chi(D)} := \begin{cases} -3\chi(D) & \text{if } \chi(D) < 0 \\ 1 & \text{if } \chi(D) = 0 \end{cases} \tag{3.1}$$

$\sim$-equivalence classes of crossings. If D is a generating diagram and has $n(D)$ link components, then

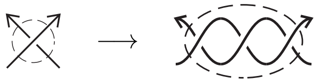
$$c(D) \leqslant \begin{cases} 4 & \text{if } \chi(D) = -1 \text{ and } n(D) = 1 \\ 2 & \text{if } \chi(D) = 0 \\ -6\chi(D) & \text{if } \chi(D) < 0 \text{ and } n(D) = 2 - \chi(D) \\ -5\chi(D) + n(D) - 3 & \text{otherwise.} \end{cases} \tag{3.2}$$

We call a generator D *maximal*, if (3.1) is an equality for D. It is known, essentially from [65] (in the case of knots, but for links the arguments are the same), that maximal generators are special alternating. We will discuss these generators in more detail in Section 5.4. See also the remark in Section 2.6.

3.2. Braiding sequences

Let D be an oriented link diagram. For each crossing in D, there is a local move, which are call $\bar{t}'_2$, and is shown on (3.3). (Note that the strand orientation at the crossing is essential.) A $\bar{t}'_2$ move will sometimes be called a *reverse* or *antiparallel twist*.

(3.3)



Let D be an oriented link diagram, with crossings $c_1, \ldots, c_n$. We explain now, following [46], how to define a family of diagrams $\mathcal{D} = \mathcal{B}(D)$ called the *braiding sequence* (or *series*) associated to D. Braiding sequences will be of importance to us for reasons we will shortly explain.

We define the braiding sequence $\mathcal{D} = \mathcal{B}(D)$ as follows. Consider the family of diagrams

$$\mathcal{D} = \{D(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{Z} \text{ odd}\}\,. \tag{3.4}$$

Herein the diagram $D(x_1, \ldots, x_n)$ is obtained from D by replacing the crossing c_i by a tangle consisting of $|x_i|$ reverse half-twists of sign $\mathrm{sgn}(x_i)$:

(3.5)

$x_i = -3$ $\qquad$ $x_i = -1$ $\qquad$ $x_i = 1$ $\qquad$ $x_i = 3$

In terms of $\bar{t}'_2$ twists, one can describe the above tangles as follows: we keep or switch the crossing c_i so that is has sign $\mathrm{sgn}(x_i)$, and apply

$$\widetilde{x}_i := (|x_i| - 1)/2 \tag{3.6}$$

times a $\bar{t}'_2$ twist on it. Thus $\mathcal{B}(D)$ does not in fact depend on how crossings in D are switched. In particular, we can assume w.l.o.g. that D is alternating.

Note that, so far, some parameters x_i in (3.4) may be redundant, in the following sense. If two crossings c_i and c_j are $\sim$-equivalent, then $\bar{t}'_2$ twists on them give the same diagram up to flypes. Thus the two tangles in (3.5) parametrized by x_i and x_j can be combined (after flypes) into a single tangle, set to be parametrized by $x_i + x_j$. This allows one then to choose $n = t(D)$ in (3.4), when lifting the constraint that x_i be odd.

3.3. Decomposition of the set of diagrams of given genus

After this simplification, we can assume a braiding sequence $\mathcal{B}(D)$ for a generating diagram D parametrized by $t(D)$ numbers x_i (cf. Definition 3.3).

Now x_i is odd if the $\sim$-equivalence class of crossing c_i is trivial, and $x_i \neq 0$ is even otherwise. (For $x_i = 0$ we have a trivial clasp, up to flypes.)

A reverse twist does not change the canonical genus: when D' is obtained from D by a $\bar{t}_2'$ twist, then $g(D') = g(D)$. Thus $g(D') = g(D)$ is constant for all $D' \in \mathcal{B}(D)$. As it turns out, some kind of converse of this property is true for fixed $g(D)$, up to finite indeterminacy. Here is the precise statement.

Theorem 3.5 (see [47]). — *The set of knot diagrams on which the Seifert algorithm gives a surface of given genus, regarded up to crossing changes and flypes, decomposes into a finite number of reverse braiding sequences $\mathcal{B}_i = \mathcal{B}(D_i)$. The same is true for link diagrams of fixed number of components.*

The Theorem 3.5 was proved in [47] for knots, but it had been briefly previously observed also by Brittenham (unpublished). His argument could be applied to links, too, although they did not receive any treatment. At the latest, the situation for links should have been sufficiently clarified in [62] with Theorem 3.4, stated above.

The investigation of how to choose a minimal set $\{D_i\}$ led to the already introduced notion of a (genus) generator. We can assume that D_i are alternating. Moreover, it can be seen that whenever D' is obtained from D by $\bar{t}_2'$ twists (and crossing changes), then, after the explained exclusion of trivial clasp diagrams, $\mathcal{B}(D') \subset \mathcal{B}(D)$. Thus we consider alternating diagrams D which cannot be obtained from diagrams D' of smaller crossing number by $\bar{t}_2'$ twists and flypes. Such diagrams D are the generating diagrams of Definition 3.3. They are those occurring as D_i in Theorem 3.5. Their underlying (alternating) knots K_i are the *generators* of genus $g = g(D_i)$. (Theorem 2.2 holds analogously for alternating knots, and thus indeed $g(K_i) = g(D_i)$.) Sometimes it is better to work with the *positive generating diagrams*, i.e., the positifications (recall Section 2.3) of D_i.

There are systematical ways to determine the generator sets $\{K_i\}$ for small g. The case $g = 1$ was done by hand in [47] (and observed independently in [45]), and $g = 2, 3$ in [57], already using substantial computation. For $g = 4$, the limit of the feasible, an account is given separately [62]. The generator sets quickly become highly difficult, and each new set required an increasingly efficient algorithm to determine.

4. Coefficients of the Conway polynomial

We prove some properties of the coefficients $\nabla_k = [\nabla]_{z^k}$ (cf. (2.9)) of the Conway polynomial on positive links. (Here it is essential to consider several components.) Some of them are needed in the following applications, but deserve interest in their own right.

4.1. Lower bounds

The Conway polynomial of a positive braid knot was studied by van Buskirk [9], who obtained inequalities for the coefficients of ∇. Later Cromwell [12, Corollary 2.1] showed that for a positive or almost positive link ∇ is positive (that is, all its coefficients are non-negative; recall Definition 2.1). Our first improvement of this result is as follows.

PROPOSITION 4.1. — *If L is a non-split positive n-component link, then for $n-1 < k \leqslant 1-\chi(L)$ and $n-k$ odd,*

$$[\nabla(L)]_{z^k} \geqslant \frac{3-\chi-k}{2}. \tag{4.1}$$

Also $[\nabla(L)]_{z^{n-1}} > 0$. (In particular, $\nabla(L)$ is 2-strictly positive.)

Remark 4.2. — Keep in mind the degrees of ∇ stated in (2.18). (The leftmost and rightmost inequalities are equalities here.) We noted that for knots $\nabla_0 = 1$, and for links ∇_{n-1} is expressible by linking numbers. Thus one cannot expect increasing growth. We study the case $k = n-1$ in the next subsection.

For the next arguments a dichotomy of crossings must be set up.

DEFINITION 4.3. — *Call a crossing in a link diagram* mixed *if it involves strands of different components, and* pure *otherwise.*

Proof of Proposition 4.1. — Consider first (4.1). We use induction on the number of crossings of a positive diagram D of L. If $n > 1$, we apply the Conway skein relation (2.8) at a mixed crossing of D (which exists since D is not split). Then D_0 has $n-1$ components, and (4.1) is inherited from the $\nabla(D_0)$ term in (2.8) (and the property of Cromwell that $\nabla(D_-)$ is positive). In case $n = 1$, i.e., L is a knot, we apply (2.8) at some non-nugatory crossing of D. Then the $\nabla(D_0)$ term proves (4.1) except if $k = 2$. In that case the inequality we require reads $\nabla_2(L) \geqslant g(L)$. This inequality was proved in [51].

The claim

$$[\nabla(L)]_{z^{n-1}} > 0$$

follows with the same argument, only using $\nabla_0(L) = 1$ for a knot L (instead of $\nabla_2(L) \geqslant g(L)$). ☐

Lemma 4.4. — *Let D be a positive n-component link diagram. Then on the series it generates by $\bar{t}'_2$ moves any coefficient ∇_k with $n-1 < k \leqslant 1-\chi(D)$ and $k-n$ odd grows unboundedly. (That is, $\nabla_k(D_i) \to \infty$ on any infinite sequence of diagrams (D_i) in this series.) The same property is enjoyed by ∇_{n-1} if all crossings of D are mixed.*

Proof. — Let $D(x_1,\dots,x_t)$ be a positive diagram in the series of a positive generating diagram D, as defined in (3.5). Thus each of the $t = t(D)$ classes of D is switched to be positive (recall Definition 3.3), and at some crossing of class i of parity $\widehat{x}_i \in \{1,2\}$

$$\widetilde{x}_i = \frac{x_i - \widehat{x}_i}{2}$$

$\bar{t}'_2$ moves (3.3) are applied (compare with (3.6)).

We use the Conway skein relation (2.8) for

$$D_+ = D(x_1,\dots,x_t) \quad \text{and} \quad D_- = D(x_1,\dots,x_{i-1},x_i-2,x_{i+1},\dots,x_t).$$

For D_0 we apply Proposition 4.1 (in fact, we use here only that the right hand-side of (4.1) is positive). We obtain then for

$$\nabla_k(x_1,\dots,x_t) := [\nabla(D(x_1,\dots,x_t))]_{z^k}$$

and for any $1 \leqslant i \leqslant t$ with $x_i > 1$ that

$$\nabla_k(x_1,\dots,x_t) \geqslant \nabla_k(x_1,\dots,x_{i-1},x_i-2,x_{i+1},\dots,x_t) + 1. \tag{4.2}$$

Note that for the inequality (4.2) we need that either $n-1 < k$ or that the diagram obtained by smoothing the twisted crossing has one component less than D, i.e., the crossing we twist at is mixed. By applying (4.2) inductively, we obtain

$$\nabla_k(x_1,\dots,x_t) \geqslant \nabla_k(\widehat{x}_1,\dots,\widehat{x}_t) + \sum_{i=1}^{t}\widetilde{x}_i = [\nabla(D)]_{z^k} + \sum_{i=1}^{t}\widetilde{x}_i. \tag{4.3}$$

The claim follows. □

Theorem 4.5. — *If L is a positive non-split link with a positive reduced diagram D of at least 5 crossings, and $1+n(L) \leqslant j \leqslant 1-\chi(L)$ with $j-n(L)$ odd, then*

$$\nabla_j(L) \geqslant \frac{9}{5} + \frac{c(D)-n(D)}{10} - \frac{j}{2}. \tag{4.4}$$

Proof. — The claimed inequality comes from balancing two estimates. First we have (4.1), which we will come back later to.

To produce a counterpart to (4.1), we use (4.2) (under change of notation here). It says that if a positive diagram D is obtained from a positive diagram D' by a (positive) $\bar{t}'_2$ move, then

$$\nabla_j(D) - \nabla_j(D') \geqslant \frac{c(D) - c(D')}{2}. \tag{4.5}$$

The r.h.s. is equal to 1 here, but we wrote the inequality in this form, because we see that then the argument can be iterated. Thus (4.5) remains true when D is obtained from D' by a sequence of $\bar{t}'_2$ moves, and there is no problem in also adding flypes.

Then we can set $D' = \widetilde{D}$ in (4.5) to be the positive generating diagram in whose series D lies. Thus we have

$$\nabla_j(D) \geqslant \frac{c(D) - c(\widetilde{D})}{2} + \nabla_j(\widetilde{D}). \tag{4.6}$$

Next, we use the estimate in Theorem 3.4, with the notice that $\chi(\widetilde{D}) = \chi(D)$ and $n(\widetilde{D}) = n(D)$:

$$c(\widetilde{D}) \leqslant -5\chi(D) + n(D) - 3. \tag{4.7}$$

To apply this main case, let us argue how to get disposed of the three other (exceptional) cases.

If $n = -\chi = 1$, the case $\widetilde{D}$ is the positivized figure-8 knot diagram, we need to consider only $j = 2$, and the inequality (4.4) to justify becomes

$$\nabla_2(D) \geqslant \frac{c(D) + 7}{10}.$$

This is true for $c(D) \geqslant 5$, for example, from the inequality

$$\nabla_2(D) \geqslant \frac{c(D)}{4}, \tag{4.8}$$

proved for knots in [51]. (The conclusion (4.4) is indeed false when $c(D) = 4$, since then K is the trefoil, and $\nabla_2 = 1$.)

The exception to (4.7) for $\chi = 0$, i.e., $\widetilde{D}$ being the Hopf link diagram, has genus 0, as has the third case $n = 2 - \chi$. For genus 0, there is no applicable j in (4.4), i.e., nothing is claimed. Thus we can assume (4.7).

After applying (4.7) to $-c(\widetilde{D})$ in (4.6), we can apply (4.1) to $\nabla_j(\widetilde{D})$ with $\chi(\widetilde{D}) = \chi(D) = \chi(L)$. Then we have

$$\begin{aligned} \nabla_j(D) &\geqslant \frac{c(D)}{2} + \frac{5\chi(D) - n(D) + 3}{2} + \frac{3 - \chi(D) - j}{2} \\ &= \frac{c(D) + 6 - n(D) - j + 4\chi(D)}{2}. \end{aligned} \tag{4.9}$$

Now we have two alternative (lower) estimates for $\nabla_j(D)$, given by (4.1) (applied directly on D) and (4.9), in which χ appears with opposite sign. The weakest choice over varying χ for given $c(D)$, $n(D)$, and j appears when either estimates are equal. The result in (4.4) follows by evaluating these two estimates for the value of χ equalizing them. □

The linear lower bound of (4.4) was motivated by, and aims at, extending the inequality (4.8) of [51] for knots. One can argue that, up to the improvement of the constant $\frac{1}{10}$, a better than linear lower bound is not possible for general positive diagrams.

On the opposite side, we observe that, by the result in [37] (see the remark below (2.15)), the inequality (4.4) for $n-1 < j = 1-\chi$ can be extended to semihomogoneous diagrams.

Corollary 4.6. — *If L is a semihomogeneous non-split link of positive genus with a semihomogeneous reduced diagram D of at least 5 crossings, then*

$$\left|\nabla_{1-\chi}(L)\right| \geqslant \frac{13 + c(D) - n(D) + 5\chi(D)}{10}.$$

4.2. Linking numbers

Now we complete the discussion of inequalities of the sort of Theorem 4.5, with the remaining case $j = n-1$ in (4.4). The coefficient $\nabla_{n-1}(L)$ is fully determined by the linking numbers $lk(L_i, L_j) = l_{ij}$ of a the components L_i of a link $L = \sqcup_{i=1}^n L_i$ from the formula of Hoste [22] and Hosokawa [21].

When all $l_{ij} \geqslant 0$ (as for a positive link L), one can state the formula thus. Associate to L a graph $G = G(L)$, the *linking graph*. The vertices of G are labeled by j for the component L_j of L. If $l_{ij} > 0$, an edge with multiplicity l_{ij} connects in G the vertices i and j. Then a positive link L is non-split iff G is connected. The Hoste–Hosokawa formula claims that then $\nabla_{n-1}(L)$ is equal to the number of spanning trees of G.

Recall Definition 4.3. It follows from the description of ∇_{n-1} that a $\bar{t}'_2$ move will not alter this coefficient if the crossings involved in the move are pure. Thus there is no estimate like (4.4) for $j = n-1$. However, when $g(L) = 0$ (the case that no j are applicable in (4.4)), all crossings in D are mixed. (Smoothing out a pure crossing would otherwise give a diagram of negative genus.) Then indeed (4.4) holds (for $j = n-1$, when $n \geqslant 2$ and $c(D) \geqslant 4$), as a consequence of the following better estimate.

Proposition 4.7. — *If L is a positive non-split link of genus* 0 *and* $n \geqslant 2$ *components, with a positive reduced diagram D, then*

$$\nabla_{n-1}(L) \geqslant \frac{c(D)}{2} - n(D) + 2. \tag{4.10}$$

Note that for such D, the number of vertices of $G(L)$ is n, and since all crossings in D are mixed and positive, the number of edges (multiple edges counted by muliplicity) is $c(D)/2$. Thus Proposition 4.7 mainly rephrases the below statement about (multi)graphs.

Lemma 4.8. — *The number of spanning trees of a connected multigraph G with $v \geqslant 2$ vertices and e edges is at least $e - v + 2$, and this bound is sharp.*

Proof. — Use induction on v. For $v = 2$ the claim is obvious. When $v > 2$, consider an edge of minimal positive multiplicity

$$1 \leqslant a \leqslant \frac{2e}{v(v-1)}. \tag{4.11}$$

Then contraction of this edge, and induction on the contracted graph yields at least

$$a(e - v + 3 - a)$$

spaning trees in G. It is easy to see that this is at least $e - v + 2$, unless $a \leqslant 0$ (which we excluded) or $a \geqslant e - v + 3$. However, the latter option also fails, because of the right inequality in (4.11) (for $e \geqslant v - 1$ by connectivity, and $v \geqslant 2$).

To see finally that the stated bound is sharp, consider G being a (spanning) tree with exactly one of its edges made multiple. □

Corollary 4.9. — *If L is a positive non-split n component link having a positive reduced diagram D with $c(D) \geqslant 5$, then*

$$\max(\nabla_{n-1}(L), \nabla_{n+1}(L)) \geqslant \frac{13 + c(D) - 6n(D)}{10}.$$

Proof. — Combine (4.4) for $j = n + 1$ when $g(L) > 0$ with (4.10) when $g(L) = 0$ (using that $c \geqslant 2(n-1)$, and $n > 1$ for $g(L) = 0$). □

4.3. Further conditions for the Conway polynomial

We now combine the previous inequalities with a reverse estimate in [58] (for arbitrary links) to obtain further new conditions on the Conway polynomial of positive, and in part more generally of semihomogeneous, links.

Proposition 4.10. — *The Conway polynomial of a positive non-split link L, which is not the unknot and the trefoil, satisfies for* $\max(n(L)-1,1)\leqslant i\leqslant 1-\chi(L)$, *and* $n(L)+1\leqslant j\leqslant 1-\chi(L)$ *with* $j-n(L)$ *odd the inequality*

$$\nabla_i(L)\leqslant \frac{1}{2}\binom{10\nabla_j(L)+5j+n(L)-19}{i}. \tag{4.12}$$

It will be clear from the proof that here, too, the remark above Corollary 4.6 applies, and we have for a non-split semihomogeneous link L of positive genus,

$$\big|\nabla_i(L)\big|\leqslant \frac{1}{2}\binom{10|\nabla_{1-\chi(L)}(L)|-5\chi(L)+n(L)-14}{i}.$$

These inequalities are not very sharp in general, yet they display a noteworthy consequence: *all* coefficients of ∇ for a positive (and partly some of a semihomogeneous) link are *directly* interrelated in complicated ways. Some improvement of the constants seems possible, but not straightforward: at least in simple cases, we will observe them below hitting the boundary of the correct.

Proof. — We will first use the following inequality, proved in [58, Lemma 4.1]. The maximal bridge length $d(D)$ of a link diagram D is the maximal number of consecutive crossing overpasses (or underpasses) of any component of D. This quantity was introduced by Kidwell [28].

If D is a link diagram of $c=c(D)$ crossings, maximal bridge length $d=d(D)$, then for $k>0$,

$$|\nabla_k(D)|\leqslant \frac{1}{2}\binom{c-d+1}{k}. \tag{4.13}$$

The proof uses the techniques of Kidwell and those in [50]. It is essentially the observation that the skein Relation (2.8) for ∇ fits with the triangular identity (and monotonicity) of binomial coefficients.

We use (4.13) to observe first for a positive diagram D that, unless all $\sim$-equivalence classes of D are trivial (cf. Definition 3.3), we have

$$\nabla_i(D)\leqslant \frac{1}{2}\binom{c-1}{i}. \tag{4.14}$$

(Keep in mind that $\nabla_i(D)\geqslant 0$, and $\nabla_i(D)=0$ for $i-n(D)$ even.) To justify (4.14), observe first that whenever D is non-alternating (i.e., not special alternating), $d(D)\geqslant 2$, and (4.14) follows directly from (4.13).

If D is alternating and has a non-trivial $\sim$-equivalence class, consider the skein relation of ∇ at a crossing p in that class. We have for alternating

$D = D_+$, that $d(D_-) = 3$ (unless D is a Hopf link diagram, which we can safely ignore). Also, crossings $\sim$-equivalent to p become nugatory in D_0, and thus we have after reducing $c(D_0) \leqslant c(D) - 2$. Again using the triangular identity of binomial coefficients in (4.13) for $\nabla_i(D_-)$ and $\nabla_{i-1}(D_0)$ gives (4.14).

Now, from Theorem 4.5 we obtain

$$c(D) \leqslant 10\nabla_j(L) + 5j + n(L) - 18, \tag{4.15}$$

and then we can use (4.14) to deduce (4.12).

It remains to check the case that all $\sim$-equivalence classes of D are trivial, i.e., $t(D) = c(D)$. Here, with (4.13) (and $d \geqslant 1$), it is still enough to have instead of (4.15),

$$c(D) \leqslant 10\nabla_j(L) + 5j + n(L) - 19. \tag{4.16}$$

To justify this, we use Theorem 3.4. If $\chi(D) = 0$, there are no i, j for which we claim (4.12). Otherwise, by (3.1), we have $c(D) = t(D) \leqslant -3\chi(D)$. Then by (4.1), we obtain

$$c(D) \leqslant -3\chi(D) \leqslant 6\nabla_j(D) + 3j - 9.$$

This will fail to imply (4.16) only if

$$4\nabla_j(D) + 2j + n(D) < 10. \tag{4.17}$$

Since $\nabla_j(D) \geqslant 1$ and $j \geqslant 1 + n(D) \geqslant 2$, we see that the only case (4.17) holds is when $n(D) = 1$ (i.e., we have knots), $j = 2$ and $\nabla_2 = 1$. Then, for example from (4.8), we see that the only exception occurs for L being the trefoil. (Note, that then indeed (4.12) is false, for the only interesting values $i = j = 2$.) □

Remark 4.11. — For L special alternating, a series of further inequalities comes from a property explained in [53], namely, that all zeros of ∇ (are real and) lie in $[-4, 0]$. It is known (as discussed in [53]) that coefficients of polynomials with all zeros real are log-concave. By using $\nabla_0 = 1$ for knots, we have then, for example, $\nabla_i^j \geqslant \nabla_j^i$ for $i \leqslant j$. This relation is not (generally) stronger than (4.12). Again for (special) alternating links, there are further inequalities imposing some global (i.e., relating all ∇_j together) constraints. These include Crowell–Murasugi's alternation property of the coefficients of Δ, and some inequalities found by Ozsváth–Szabó for knots (see [62]).

5. Link polynomial degrees

5.1. Unbounded growth of skein polynomial degrees

In this section we will combine the results of Section 4, notably Theorem 4.5, with the main result in [61].

THEOREM 5.1 ([61]). — *All coefficients (for fixed degree in all variables) of V, P and F are bounded (that is, admit only finitely many values) on positive links.*

While this theorem was proved in [61] mostly with focus on knots, dealing with links requires extra arguments at this point, which are useful to be pointed out and be taken care of.

We will use below $\chi(L)$ for the Euler characteristic and $n(L)$ for the number of components of a positive/negative link L. By D we will usually denote a positive/negative reduced diagram of L. We remind that $\chi(L) = \chi(D)$. Moreover, when L is normal,

$$c(L) \geqslant c(D) + \chi(L), \tag{5.1}$$

as proved in [60].

PROPOSITION 5.2. — *Let (L_i) be a sequence of (pairwise distinct) positive non-split links. Let $k \in \mathbb{N}$ be fixed so that $k - n(L_i)$ is odd for all i, and one of the following conditions holds: either*

$$n(L_i) - 1 < k \leqslant 1 - \chi(L_i), \tag{5.2}$$

or

$$k+1 = n(L_i) \in \{1, 2-\chi(L_i)\} \quad \text{(i.e., } L_i \text{ are knots, or links of genus } 0). \tag{5.3}$$

Then as $i \to \infty$,

$$\max\deg_l [P(L_i)]_{m^k} \longrightarrow \infty. \tag{5.4}$$

Proof. — By a standard subsequence argument, it suffices to assume that either $\chi(L_i)$ are bounded below, or $\chi(L_i) \to -\infty$. In latter case we have $[P(L_i)]_{m^k} \neq 0$ because of (2.7) and $\nabla_k > 0$ in Proposition 4.1. Then

$$\max\deg_l [P(L_i)]_{m^k} \geqslant \min\deg_l [P(L_i)]_{m^k} \geqslant \min\deg_l P(L_i) = 1 - \chi(L_i),$$

and the assertion follows.

Thus assume $\chi(L_i)$ are bounded below. In case of (5.2), use Theorem 5.1 for the HOMFLY–PT polynomial and (4.4). If (5.3), for genus 0 links use (4.10). For knots and $k = 0$ use that one can express $v_2 = \nabla_2$ by different means from $[P]_{m^0}$. □

Remark 5.3. — For knots, using the relation between v_2 and $[P]_{m^k}$ for $k = 0, 2$, and the Gauß diagram sum expression for v_2, we showed in [52, Corollary 5.5 and its proof] the growth result (5.4) more generally for almost positive knots. It would be interesting to extend the above argument to almost positive knots for any (applicable) k.

Corollary 5.4. — *Let (L_i) be a sequence of (pairwise distinct and possibly split) positive links. Then* $\max\deg_l P(L_i) \to \infty$.

Proof. — Proposition 5.2 clearly shows this if L_i are non-split. For split L_i, use (2.12). □

The following corollary answers [54, Question 5.4] in the expected positive way and extends Theorem 5.2 therein.

Corollary 5.5. — *Only finitely many s-almost positive normal links are negative.*

Proof. — One easily observes that $\chi(D) = \chi(L) < 0$ for any positive diagram D of a normal link L, so that $\min\deg_l P(L) = 1-\chi(L) > 0$. Then for s-almost positive link diagrams D, we have $\min\deg_l P(D) \geqslant 1 - 2s$ by [34]. Proposition 5.4 implies, however, that $\min\deg_l P \to -\infty$ on any infinite sequence of negative links. □

The easy implication to Corollary 5.5 is the main reason for formulating Corollary 5.4. However, by using a more careful argument, we can considerably sharen this assertion.

5.2. Estimates of skein polynomial degrees

Assume first L is non-split, and D is a positive (connected) reduced diagram. Consider the leading m-coefficient (2.14) $\widetilde{P}(D) = [P(L)]_{m^{1-\chi(L)}}$ of $P(D)$. Then

$$\min\deg_l \widetilde{P}(D) = 1-\chi(D) = 1-\chi(L), \tag{5.5}$$

because $\max\deg_m P(L) = \min\deg_l P(L) = 1-\chi(L)$ and the identity [31, Proposition 21]. (H. Morton remarked this to me.)

We claim the following inequality, which then easily leads to Theorems 1.1 and 1.4.

Theorem 5.6. — *If D has no unknot split components and is semihomogeneous,*

$$\operatorname{span}_l P(D) \geqslant c(D) - 1 + 4\chi(D). \tag{5.6}$$

Proof. — By (2.13) (when Hopf link split components are involved) and its analogue for $\widetilde{P}$ (otherwise), it is enough to see the inequality

$$\operatorname{span}_l \widetilde{P}(D) \geqslant c(D) - 1 + 4\chi(D), \tag{5.7}$$

for a connected (non-trivial and non-Hopf link) diagram D, where $\chi(D) \leqslant 0$.

Thus assume D is connected. The special case $\chi = 0$ can be handled (ad hoc). Then $L = \overline{T}_{2,p}$ is a reverse $(2, p)$-torus link (p even), and D is its (unique) minimal (p) crossing diagram. The skein polynomial of such a link is well-known, and

$$\operatorname{span}_l \widetilde{P}(\overline{T}_{2,p}) = p - 2 = c(D) - 2. \tag{5.8}$$

(For $p = 2$, use $\operatorname{span}_l P(\overline{T}_{2,2}) = 2$.)

Thus we assume $\chi(D) < 0$. Let us argue next why it is enough to prove (5.7) for positive D, rather than semihomogeneous. This is essentially by the multiplicativity of $\widetilde{P}$ under (diagrammatic) Murasugi sum in (2.15).

Let

$$D = D_1 * \cdots * D_n * D'_1 * D'_2 * \cdots * D'_{n'} \tag{5.9}$$

be the Murasugi atom decomposition of a semihomogeneous diagram D of a link L into positive atoms D_i and negative ones D'_j. Let $\widehat{D}$ be the positification of D and $\widehat{L}$ its link. Then

$$\widehat{D} = D_1 * \cdots * D_n * !D'_1 * !D'_2 * \cdots * !D'_{n'}.$$

From the mirroring property

$$P(D'_j)(l, m) = P(!D'_j)(l^{-1}, m)$$

of P we have

$$\operatorname{span}_l \widetilde{P}(D'_j)(l) = \operatorname{span}_l \widetilde{P}(D'_j)(l^{-1}) = \operatorname{span}_l \widetilde{P}(!D'_j)(l). \tag{5.10}$$

Now because of (2.15), the additivity of span_l under multiplication, and (5.10), we have $\operatorname{span}_l \widetilde{P}(D) = \operatorname{span}_l \widetilde{P}(\widehat{D})$. Moreover, $\chi(L) = \chi(D) = \chi(\widehat{D}) = \chi(\widehat{L})$ and $c(D) = c(\widehat{D})$.

Thus we will assume that D is (connected and) positive.

In the following we will be looking at reverse clasps C in D. Note that in the Seifert circle picture of D, a C gives a (valence-two) Seifert circle. We call the two other Seifert circles near the crossings of C *external*. We call C *good* if its external Seifert circles are distinct, and *bad* otherwise (i.e., when there is actually only one external Seifert circle).

Lemma 5.7. — *Let D be a positive diagram having a good reverse clasp C and let D' be the positive diagram obtained from D by resolving C. Then $\chi(D') = \chi(D)$, and if D is a generating diagram, so is D'.*

Proof. — We have $c(D') = c(D) - 2$, and because C is good, $s(D') = s(D) - 2$. It is easy to see that if resolving C makes in D' crossings $\sim$-equivalent which were not $\sim$-equivalent in D, then C must be bad. See [57, Proof of Lemma 2.1, and in particular Figure 4]. □

Lemma 5.8. — *Under the same assumptions as Lemma 5.7, we have*

$$\max\deg_l \widetilde{P}(D) \geqslant \max\deg_l \widetilde{P}(D') + 2. \tag{5.11}$$

Proof. — One can apply the skein Relation (2.2) for P at a crossing in C (with $D_+ = D$ and $D_- = D'$). Then it is enough to see that the leading l-term of $-l^2\widetilde{P}(D')$ is not cancelled from the one of $-l\widetilde{P}(D_0)$. In fact, no coefficients of these two polynomials cancel at all, and this follows from a positivity property of $\widetilde{P}$, which we crucially use here. In was observed in [12], that the existence of a positive skein resolution tree for a positive (connected) diagram $\widetilde{D}$ implies that $\widetilde{P}(\widetilde{D})$, after a suitable change of variables (to v, z, as used there), has only positive coefficients. □

Note in particular that clasps created by a $\bar{t}_2'$ move are always good. The consequence of the last lemma will be conveniently rewritten thus. Let

$$\lambda(D) = c(D) - \operatorname{span}_l \widetilde{P}(D).$$

Corollary 5.9. — *If $D \in \mathcal{B}(\widetilde{D})$ for a generating diagram $\widetilde{D}$, then $\lambda(D) \leqslant \lambda(\widetilde{D})$.*

This means that for (5.7) it is enough to prove

$$\lambda(D) \leqslant 1 - 4\chi(D) \tag{5.12}$$

for a generating diagram D. We assume thus below D is generating.

Assume w.l.o.g. that D is flyped so that $\sim$-equivalent crossings are twist equivalent (i.e., we have as many reverse clasps as possible).

Lemma 5.10. — *At most $1 - \chi(D)$ reverse clasps in D are bad.*

Proof. — Each bad reverse clasp corresponds to a (diagrammatic) Hopf plumbing, so we need just to count the Euler characteristic. □

A combination of Lemmas 5.7, 5.8 and 5.10 now easily leads to the following. Let $t_1 = t_1(D)$ resp. $t_2 = t_2(D)$ be the number of trivial resp. non-trivial $\sim$-equivalence classes of D.

Lemma 5.11.

$$\operatorname{span}_l \widetilde{P}(D) \geqslant 2(t_2(D) - 1 + \chi(D)) \tag{5.13}$$

Proof. — Apply (5.11) repeatedly as long as you find (and resolve) any good reverse clasp. This must occur at least $t_2(D) - 1 + \chi(D)$ times by Lemma 5.10, while Lemma 5.7 ascetrains that during the iteration nothing goes wrong. □

From here we easily get to (5.6).

$$\operatorname{span}_l P(D) \geqslant c(D) - 1 + 4\chi(D) \tag{5.6}$$

We have, for $\chi = \chi(D) < 0$,

$$t_1 + t_2 = t(D) \leqslant -3\chi \tag{5.14}$$

from (3.1). For

$$t_2 \geqslant 1 - \chi,$$

implying

$$t_1 \leqslant -2\chi - 1, \tag{5.15}$$

we use (5.13). Thus

$$\lambda(D) = c(D) - \operatorname{span}_l \widetilde{P}(D) = t_1 + 2t_2 - \operatorname{span}_l \widetilde{P}(D) \leqslant t_1 + 2 - 2\chi \leqslant 1 - 4\chi$$

(in the last inequality, we used (5.15)). If $t_2 < 1 - \chi$, then also using (5.14),

$$\lambda(D) \leqslant c(D) = t_1 + 2t_2 \leqslant -3\chi + t_2 < 1 - 4\chi.$$

This shows (5.12), and hence (5.7), and proves Theorem 5.6. □

5.3. Braid index and Thurston–Bennequin number

A further use of P is to estimate the braid index $b(L)$ of a link L (see Section 2.3). Birman and Menasco proved in [6] that there are only finitely many links L of given $\chi(L)$ and $b(L)$. We can make this statement more explicit for semihomogeneous links L. (See [52, end of Section 5] for a version for almost positive *knots*.)

Proof of Theorem 1.1. — Apply the braid index inequality (2.16) of Morton–Williams–Franks [17, 34],

$$2b(L) - 2 \geqslant \operatorname{span}_l P(D) \geqslant \operatorname{span}_l \widetilde{P}(D), \text{ to (5.6).}$$ □

The estimate of the l-span now also yields the other major application.

Recall that the Thurston–Bennequin number $tb(\mathcal{K})$ of a Legendrian knot $\mathcal{K}$ in the standard contact space $(\mathbb{R}^3(x,y,z),\ \mathrm{d}x + y\,\mathrm{d}z)$ is the linking number of $\mathcal{K}$ with $\mathcal{K}'$, where $\mathcal{K}'$ is obtained from $\mathcal{K}$ by a push-forward along a vector field transverse to the (hyperplanes of the) contact structure. The Maslov (rotation) index $\mu(\mathcal{K})$ of $\mathcal{K}$ is the degree of the map

$$t \in S^1 \mapsto \frac{\mathrm{pr}\,\frac{\partial \mathcal{K}}{\partial t}(t)}{\left|\mathrm{pr}\,\frac{\partial \mathcal{K}}{\partial t}(t)\right|} \in S^1,$$

where $\mathrm{pr} : \mathbb{R}^3 \to \mathbb{R}^2 \simeq \mathbb{C}$ is the projection $(x,y,z) \mapsto (y,z)$. For links the invariants are obtained by summing over all components.

Our general statement in [55], which extended the result of [24], implied in particular that $tb(\mathcal{K})+|\mu(\mathcal{K})|$ for a Legendrian embedding $\mathcal{K}$ of a negative knot K becomes arbitrarily small when the crossing number of K increases. We obtain now the more explicit estimate (also for links) stated above.

Proof of Theorem 1.4. — We use the inequalities of [19, Section 2, Theorem 2.4] for tb and μ coming from $\min\deg_l P$. We have, with $!L$ a positive link,

$$tb(\mathcal{L}) + |\mu(\mathcal{L})| \leqslant \min\deg_l P(L) - 1 = -\max\deg_l P(!L) - 1. \tag{5.16}$$

Thus it is enough to prove for a *positive* link L with a positive reduced diagram D that

$$\max\deg_l \widetilde{P}(D) \geqslant \frac{3 + c(D)}{4}. \tag{5.17}$$

(For L being the Hopf link use $\max\deg_l P$ instead.)

By (2.12) and its analogue for $\widetilde{P}$ we see that (5.17) is preserved under split unions (also with unknots and Hopf links), as long as D is not just a single unknot (0-crossing) diagram.

Thus we prove D again for a connected (and non-trivial) diagram D. We explained how to deal with the Hopf link, and for the other links of $\chi = \chi(D) = 0$ the situation is easy, so let again $\chi < 0$. We have from (5.5) and (5.6),

$$\max\deg_l \widetilde{P}(D) = \min\deg_l \widetilde{P}(D) + \mathrm{span}_l\, \widetilde{P}(D) \geqslant c(D) + 3\chi(D).$$

A comparison between this and (5.5) itself gives the estimate

$$\max\deg_l \widetilde{P}(D) \geqslant \max(c(D) + 3\chi(D), 1 - \chi(D)).$$

Its alternatives are balanced for $\chi(D) = \frac{1-c(D)}{4}$, where the stated value on the right of (5.17) occurs. □

For a general semihomogeneous link one has a negativity result when choosing between mirror images.

Proposition 5.12. — *If L is normal and semihomogeneous with a semihomogeneous diagram D, and $\mathcal{L}'$ is a Legendrian embedding of $!L$, then*

$$\min(tb(\mathcal{L})+|\mu(\mathcal{L})|,\ tb(\mathcal{L}')+|\mu(\mathcal{L}')|)\leqslant\frac{-1-c(D)}{2}-2\chi(L).$$

Proof. — This follows from (5.16) and (5.6). □

Now we turn to Theorem 1.3. Apart from addressing semihomogeneous insead of homogeneous links, which is only a minor technicality, note at least four more impactful advantages of this theorem in comparison to [55]: it

(a) gives an explicit simple estimate,
(b) does not restrict (fix) χ,
(c) does not require the choice of mirror images, and
(d) applies to links, too.

Proof of Theorem 1.3. — It is easy to see, by additivity under split union, that it is enough to prove (1.4) for a connected diagram D.

Let again $\widehat{D}$ be the positification of D. By using (5.9) and (5.10), and applying (5.5) to $\widehat{D}$ and D'_j, we have

$$\begin{aligned}\min\deg_l\widetilde{P}(D)&=\min\deg_l\widetilde{P}(\widehat{D})-\sum_{j=1}^{n'}2\min\deg_l\widetilde{P}(!D'_j)+\operatorname{span}_l\widetilde{P}(D'_j)\\&=1-\chi(D)-\sum_{j=1}^{n'}2-2\chi(D'_j)+\operatorname{span}_l\widetilde{P}(D'_j).\end{aligned}$$

Next, we use (5.7) on D'_j with the remark that for $c(D'_j)=2$, the inequality holds with r.h.s. decreased by 1. Thus

$$1-\chi(D)-\min\deg_l\widetilde{P}(D)\geqslant\sum_{j=1}^{n'}\max(2-2\chi(D'_j),\ c(D'_j)+2\chi(D'_j)+\delta_j)$$

with δ_j being 1 for $c(D'_j) > 2$ and 0 for $c(D'_j) = 2$. The maximum in the sum is at least $\frac{c(D'_j)+2+\delta_j}{2}$. Then from (5.16)

$$\begin{aligned}-\chi(L) - tb(\mathcal{L}) - |\mu(\mathcal{L})| &\geqslant 1 - \chi(D) - \min \deg_l \widetilde{P}(D)\\ &\geqslant \sum_{j=1}^{n'} \frac{c(D'_j) + 2 + \delta_j}{2}\\ &= \frac{c_-(D)}{2} + \frac{1}{2}\sum_{j=1}^{n'}(2+\delta_j) \ \geqslant\ \frac{c_-(D)+3}{2}\end{aligned}$$

(the last inequality with the remark about $n' = 1$ and $c(D'_1) = 2$). □

Let us remark that there is a somewhat better estimate than (1.1) for an alternating diagram D, which was observed in [63] along a different line of reasoning, using the Jones polynomial. (It is used there to classify alternating knots of braid index 4.)

Theorem 5.13 ([63]). — *If D is a reduced alternating diagram, then*

$$\operatorname{span}_l P(D) \geqslant s(D) - 1 = c(D) + \chi(D) - 1. \tag{5.18}$$

Thus, from (2.16), for an alternating diagram D of a link L, we have, in analogy to (1.1),

$$b(L) \geqslant \frac{c(D)+1}{2} + \frac{\chi(D)}{2}.$$

This can be used for an improvement of some of the following inequalities, in particular, (5.16). We will need to appeal to (5.18) more substantially in Section 5.4 below.

Corollary 5.14. — *Under the assumptions of Theorem 1.1, we have*

$$c(L) \geqslant \frac{1+c(D)}{2} + \chi(L). \tag{5.19}$$

Proof. — Use (5.18) and that for each diagram D' of L, we have $s(D') \geqslant b(L)$ and $-\chi(L) \leqslant -\chi(D') = c(D') - s(D')$. □

Compare inequality (5.19) with (5.1) in the case of positive links.

5.4. Application to enumeration

Proof of Theorem 1.2. — By combining (1.1) with Ohyama's braid index inequality [39],

$$2b(L) - 2 \leqslant c(D), \tag{5.20}$$

we are left to count a set $\mathcal{C}_{b,\chi}$ of diagrams of given χ within a fixed (depending on χ, but not b) interval of crossings. Then use (3.1). This shows an upper bound of the type (2.1).

Note that we can distinguish the links of a positive fraction of the occurring diagrams in $\mathcal{C}_{b,\chi}$ by taking alternating diagrams, and using the Flyping Theorem (as done in [47, 65]). This shows that

$$\sum_{b=1}^{c} \beta_{b,\chi} \sim_c c^{-3\chi}.$$

However, for (1.2) we need to know that the braid index in a positive fraction of these diagrams behaves exactly as expected (see (5.21) below). For this we will exhibit a maximal generator for each χ whose links all have exact MWF inequality (2.16). This is the subject of Lemma 5.16 below, whose preparation and proof constitue the rest of the discussion for Theorem 1.2. □

Remark 5.15. — With Lemma 5.16 we have that in fact (1.2) holds also when counting only knots for odd χ and only 2-component links for even χ.

Moreover, maximal generators are special alternating ([65]), and thus in Theorem 1.2 "semihomogeneous" can be replaced by any of "homogeneous", "alternating", "positive" or "special alternating"; the resulting enumeration problems turn out to be all asymptotically equivalent.

It is tempting to conjecture asymptotic proportionality between the various classes for fixed number of components, or even asymptotic equality (as happens when braid index is replaced by crossing number [60]). This remains, though, a challenge out of reach with (the present state of) our technology.

Another problem would be to see if instead of 1 and 2 components one can prove Lemma 5.16 for any fixed number $n \geqslant 3$ of components (and every χ with $n-\chi$ even), but this requires further effort.

Before stating Lemma 5.16, we first describe the maximal generators on whose series we prove MWF (2.16) to be sharp. Recall the remarks at the end of Section 3.1 and the terminology of Section 2.6.

We give these maximal generators D in terms of their Seifert graph $G(D)$. It is a planar bipartite 2-3-valent graph, which is a reduced (bipartite) bisection of a 3-connected (planar) 3-valent graph G'. As such, G comes from a $\pm$ marking of the vertices of G': bisect an edge of G' exactly if it connects vertices with the same marking. It is thus enough to describe the marked graph G'.

Take two 1-2-3-valent planar trees Γ_1 and Γ_2, one with vertices marked $+$, and one with $-$. (See Figure 5.1 (a).) We assume Γ_1 and Γ_2 have the same number $k \geqslant 0$ of edges, with $\chi = -1 - k < 0$.

Then

$$k' = k + 3 = \sum_{v \in V(\Gamma_1)} 3 - \mathrm{val}(v) = \sum_{v' \in V(\Gamma_2)} 3 - \mathrm{val}(v').$$

Draw $3 - \mathrm{val}(v)$ external edges from each vertex v of Γ_1, similar for Γ_2 (as in Figure 5.1 (b)), and cyclically connect the k' edges between the Γ_i (as in Figure 5.1 (c)). This gives a 2-connected graph G'.

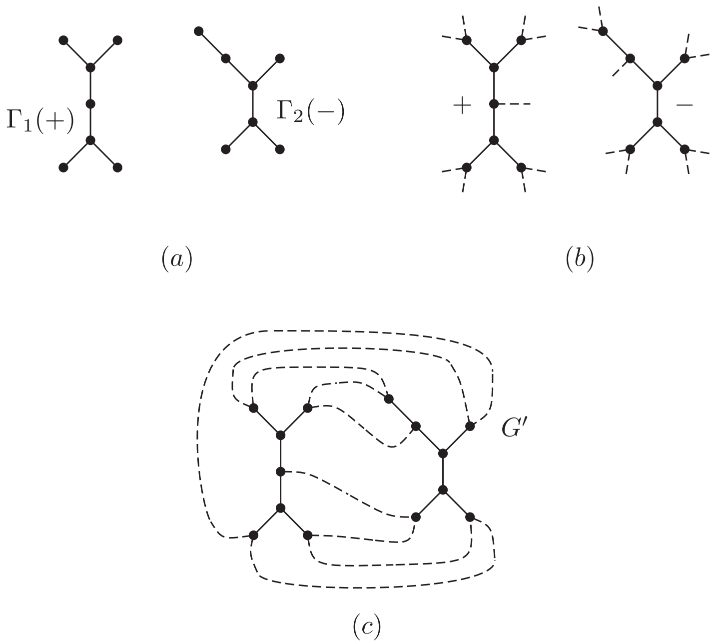


Figure 5.1. The construction of the unbisected Seifert graph of the maximal generators for Lemma 5.16. (Here $k = 6$ and $\chi = -7$, thus G' gives a knot generator of genus 4.)

If such a graph G' is 3-connected, it gives a maximal generator. This property is not automatic, however, it is easy to construct such examples for each $\chi < 0$. (Note that a 2-cut of G' may only consist of one edge in Γ_1 and one edge in Γ_2.)

The number $n = n(D)$ of components of the generator diagram $D = D_{k'}$ with unbisected Seifert graph G' is $n = 2$ for k' even and $n = 1$ for k' odd.

This can be seen as follows. Resolving clasps in D (which correspond to the edges of Γ_i bisected in G), we are left with the diagram of a $(2, k')$-torus link.

For k' (and χ) odd, one can conclude (see (5.25) below) that these generators make (3.2) sharp, and the proof of Theorem 3.4, in [62], should make clear that all maximal crossing number knot generators arise by this construction. (Note: we understood from [62] that all maximal crossing generators, except the figure-8 knot, are maximal also by number of $\sim$-equivalence classes, but the converse is clearly not true.) For $n \geqslant 2$ components, the description of maximal crossing generators is certainly more complicated.

Lemma 5.16. — *On the series $\mathcal{B}(D)$ of these generators D, MWF is sharp. Moreover,*

$$b(D') - b(D) = \frac{c(D') - c(D)}{2} \tag{5.21}$$

for all $D' \in \mathcal{B}(D)$ (where the braid index of a diagram is the braid index of its link).

Proof. — To prove this Lemma 5.16, we appeal to the work in [62] and the graph index.

The index of graphs was introduced by Murasugi–Przytycki, but for the sake of a simple definition, we use here the description of Traczyk [67].

Let G be a planar bipartite graph. We call a set S of edges of G to be *independent*, if each cycle C of length $2l$ has $|C \cap S| < l$. We say that an independent set S is *maximal independent*, if there is no independent set S' with $|S'| > |S|$. The *index* $\operatorname{Ind}(G)$ can be defined as the size $|S|$ of a maximal independent set S of G.

For a link diagram D, set $\operatorname{Ind}(D) = \operatorname{Ind}(G(D))$. Then the following holds:

Theorem 5.17. — *If D is a diagram of a link L, then*

$$b(L) \leqslant s(D) - \operatorname{Ind}(D). \tag{5.22}$$

Murasugi–Przytycki's proof of this inequality has a flaw, which was found recently. The problem was remedied partially in two papers by Traczyk [68] and myself [59], at least to the extent that (5.22) is recovered. (Certain features of Murasugi–Przytycki's method, however, go amiss.)

The combination of (5.22) with (2.16) gives

$$1 + \frac{1}{2}\operatorname{span}_l P(L) \leqslant b(L) \leqslant s(D) - \operatorname{Ind}(D). \tag{5.23}$$

Definition 5.18. — *We call D* Ind-optimal *if*

$$1+\frac{1}{2}\operatorname{span}_l P(L) = s(D) - \operatorname{Ind}(D).$$

This is a practical way to test the sharpness of MWF (and calculation of braid index). E.g., we used it in [62] to determine the braid index of alternating knots up to genus 4.

We appeal now to the following lemma, which is a special case of [62, Corollary 7.1].

Lemma 5.19 ([62]). — *Let D be an* Ind-*optimal special generator such that $G(D)$ has two disjoint maximal independent sets. Then all $D' \in \mathcal{B}(D)$ are* Ind-*optimal.*

It follows from the argument for [62, Corollary 7.1] that then also (5.21) holds for all $D' \in \mathcal{B}(D)$.

Now we argue that we can apply Lemma 5.19 on our maximal generators $D = D_{k'}$.

Lemma 5.20. — *The diagrams $D_{k'}$ satisfy the assumptions of Lemma 5.19.*

Proof. — The trees Γ_1, Γ_2 have $k = k'-3$ edges each. This gives $2(k'-3)$ edges of G' in both Γ_i, plus k' edges connecting the Γ_i. Thus G' has $3k'-6$ edges and $2(k'-2)$ vertices, which conforms to the initially stated value

$$\chi = 2 - k' = -1 - k. \tag{5.24}$$

In $G = G(D)$ the $2(k'-3)$ edges of Γ_i in G' are bisected, thus

$$c(D) = 4(k'-3) + k' = 5k' - 12 = -5\chi - 2 \tag{5.25}$$

(which matches the bound of (3.2) for χ odd and $n = 1$). These bisections also add a vertex (Seifert circle) for each edge of Γ_i. Thus

$$s(D) = \sum_{i=1}^{2} e(\Gamma_i) + v(\Gamma_i) = 2(k'-2) + 2(k'-3) = 4k' - 10, \tag{5.26}$$

and again $\chi(D) = s(D) - c(D)$ leads to (5.24), as should be.

Now, by using (5.26), Theorem 5.13, and that $\operatorname{span}_l P(D)$ is even, we have

$$\operatorname{span}_l P(D) \geqslant 4k' - 10, \tag{5.27}$$

thus by MWF (2.16)

$$b(D) \geqslant 2k' - 4. \tag{5.28}$$

On the opposite side, we have

$$\mathrm{Ind}(D) \geqslant 2(k' - 3), \tag{5.29}$$

because we can find an independent set of $G(D) = G$ of the number of clasp (valence-2) Seifert circles in D: taking one edge indicent to each one of the valence-2 vertices of $G(D)$ corresponding to these clasp Seifert circles will give an independent set S.

Now (5.27), (5.26) and (5.29), together with (5.23), combine to show on the one hand that an edge set S of the above type is maximal independent, and on the other hand that D is Ind-optimal (with braid index as in (5.28)).

Moreover, one can easily find two disjoint (maximal) independent sets S_1, S_2 by switching between the two different edges adjacent to each valence-2 vertex of $G(D)$.

This proves Lemma 5.20, and hence also Lemma 5.16. ☐

Thus the proof of Theorem 1.2 is concluded. ☐

Let us remark that, in order to obtain by the presented method asymptotic proportionality in (1.2), one would have to show that MWF in unsharp only sporadically on *every* maximal generator. This seems too hard to accomplish at the moment.

In closing, we observe that Corollary 5.14 allows for the following similar application. It extends, in some partial form, the result of [65] about the number of alternating knots and [60] about the number of positive links of fixed genus. The proof goes along the lines of the proof of Theorem 1.2.

COROLLARY 5.21. — *Let $\eta_{c,\chi}$ be the number of non-split semihomogeneous links of crossing number* at most c *and (fixed) Euler characteristic* $\chi < 0$. *Then $\eta_{c,\chi}$ satisfies the asymptotic equivalence*

$$\eta_{c,\chi} \sim_c c^{-3\chi}.$$

5.5. Degree growth for Jones and Kauffman polynomials

We can also obtain versions of Corollary 5.4 for the Jones and Kauffman polynomial, although less strong. *In this Subsection 5.5, we restrict ourselves to knots.* We also use $g(K) = g(D)$ for the genus of a knot or diagram (where $2g = 1 - \chi$). In an attempt to generalize the results to links, difficulties may occur in Corollaries 5.22 and 5.23 (because the relation between the Vassiliev invariants of ∇, V and F may require the study of linking numbers), and Proposition 5.24 (which uses Gauß sum formulas not available for links).

Corollary 5.22. — *If K_i are positive knots, then* $\max\deg V_{K_i} \to \infty$ *as* $i \to \infty$.

Proof. — Since $\max\deg V_{K_i} \geqslant \min\deg V_{K_i} = g(K_i)$ (cf. [54]), it suffices to look only on K_i of bounded, and hence w.l.o.g., fixed genus. If for such knots $\max\deg V_{K_i}$ is bounded, then Theorem 5.1 would imply that only finitely many polynomials occur as V_{K_i}. But this contradicts the growth statement in Theorem 4.5 for $-V_K''(1) = 6\nabla_2(K)$. □

Corollary 5.23. — *Let* $F_k(K) = [F(K)]_{z^k}$. *If* (K_i) *is a sequence of positive knots, then for* $0 \leqslant k \leqslant 2$ *we have*

$$\max\deg_a F_k(K_i) \to \infty. \tag{5.30}$$

Proof. — For $k = 0$, Lickorish [30, Proposition 4.7] showed that $[F(a,z)]_{z^0} = [P(a,m)]_{m^0}$ for knots (up to variable change), so that the statement we claim is implied by the case of $k = 0$ in Proposition 5.2.

For $k = 1, 2$ we use Yokota's result $\min\deg_a F = 2g$ and the relation between the Conway Vassiliev invariants $\nabla_i = [\nabla]_{z^i}$ and the Kauffman Vassiliev invariants

$$F_k^{(j)}(K) := \frac{\sqrt{-1}^{k+j}}{j!}\frac{\mathrm{d}^j}{\mathrm{d}a^j}\bigg|_{a=\sqrt{-1}} F_k(K).$$

(See [25], but note that this definition differs from that there by the first factor. We write here "$\sqrt{-1}$" for the imaginary unit to avoid conflicts with the variable i, which we mostly use for indexing.)

The Property (5.30) for $k = 1$ follows from Lemma 4.4 and the identity $F_1^{(1)} = -2\nabla_2$. Then $F_1^{(1)}(K_i) \to -\infty$, for $g(K_i)$ fixed, and by the subsequence argument as in the proof of Proposition 5.2 we are done.

Similarly for $k = 2$ it suffices to show that $F_2^{(1)}(K_i) \to \pm\infty$. Now, $F_2^{(1)}$ is Vassiliev invariant of degree $\leqslant 3$, and it its straightforward to see that it is not constant. Then the claim follows from the proposition below. □

Proposition 5.24. — *If* (K_i) *are distinct positive knots of bounded genus, and* v *is a non-constant Vassiliev invariant of degree* $\leqslant 3$, *then* $|v(K_i)| \to \infty$.

Note that for a growth result for a Vassiliev invariant v of degree $\leqslant 3$ in arbitrary form for positive knots the boundedness condition on the genus is clearly necessary; otherwise one can choose v to be primitive and to vanish on the trefoil and consider iterated connected sums of trefoils. Also, for degree 4 Proposition 5.24 is no longer true: consider ∇_4 on positive twist knots.

Proof. — By Theorem 3.5 we need to consider only a fixed braiding sequence of $\bar{t}'_2$ twists of some (genus g) diagram D, parametrized by $c(D) = n$ odd numbers $x_1, \dots, x_n$.

As v is of degree $\leqslant 3$, we write

$$\pm v = C_1 v_3 + C_2 v_2$$

with $v_2 = \nabla_2$ and v_3 being the antisymmetric (w.r.t. taking the mirror image) degree 3 Vassiliev invariant. (We can w.l.o.g. ignore a constant term.) Up to adjusting sign, we may assume $C_1 \geqslant 0$. If $C_1 = 0$, we could argue with Lemma 4.4; thus assume $C_1 > 0$.

Now we use the Gauß diagram formulas for v_2 and v_3 due to Fiedler [16] and Polyak–Viro [41]; see Section 2.5. Note that for a positive diagram, $w_p = 1$ for all crossings p. Thus all terms summed in (2.23) and (2.24) are 1, and estimating the values of v_2 and v_3 reduces to counting the number of matching configurations.

In $D(x_1, \dots, x_n)$ for all x_i odd and positive, a linked pair of crossings i and j in D, is replaced by two collections of x_i and x_j arrows, each arrow linked with any arrow from the other collection. Counting in Fiedler's formula just the (contributions from) the linked pairs and configurations $(4, 2)0$, we obtain

$$x_i \left[\binom{\frac{x_j+1}{2}}{2} + \binom{\frac{x_j-1}{2}}{2} \right] + x_j \left[\binom{\frac{x_i+1}{2}}{2} + \binom{\frac{x_i-1}{2}}{2} \right] + x_i x_j$$
$$= \frac{(x_i-1)^2}{4} x_j + \frac{(x_j-1)^2}{4} x_i + x_i x_j \geqslant \frac{x_i^2 x_j + x_j^2 x_i}{4}.$$

Hence

$$v_3(D(x_1, \dots, x_n)) \geqslant \sum_{i,\ j \text{ linked}} \frac{x_i^2 x_j + x_j^2 x_i}{4}.$$

On the other hand, by (2.23) clearly

$$v_2(D(x_1, \dots, x_n)) \leqslant \sum_{i,\ j \text{ linked}} x_i x_j.$$

Thus

$$\pm v = C_1 v_3 + C_2 v_2 \geqslant \sum_{i,\ j \text{ linked}} x_i x_j \left(\frac{C_1}{4}(x_i + x_j) + \min(C_2, 0) \right). \tag{5.31}$$

For $x_i, x_j > 0$, each summand on the right is bounded below, and grows unboundedly if so does $\max(x_i, x_j)$.

If now $D_k = D(x_{1,k}, \ldots, x_{n,k})$ are distinct diagrams in the braiding sequence of D, then

$$c(D_k) = \sum_{j=1}^{n} x_{j,k} \longrightarrow \infty$$

as $k \to \infty$, and (at least) the largest of the summands on the right of (5.31) grows unboundedly, while the others remain bounded from below. Thus $\pm v(D_k) \to \infty$, as desired. □

An alternative argument to conclude the proof of Corollary 5.23 for $k = 2$ is to remark that $F_2^{(1)}$ is in fact antisymmetric w.r.t. taking the mirror image. Thus it is a (non-trivial) multiple of v_3. We showed in [51] that $v_3 \to \infty$ on any sequence of positive knots (not necessarily of bounded genus). But now Proposition 5.24 implies more restrictive properties of $F_{0,1,2}$. For example even polynomial functions $X(F_0, F_1, F_2)$ for $X \in \mathbb{Q}[x_0, x_1, x_2]$ exhibit similar degree growth.

For F_k with $k \geqslant 3$, the evident problem is that the Vassiliev invariants it contains have higher degree, and thus are more difficult to control combinatorially. In an attempt to apply the positivity result for the Conway Vassiliev invariants, the only further relation one can use is

$$\frac{F_2^{(1)}}{2} + F_2^{(2)} - 6F_3^{(1)} = \nabla_2 - 7\nabla_2^2 + 18\nabla_4. \tag{5.32}$$

See [25, p. 422]. Here also the r.h.s. empirically appears to drop unboundedly for positive knots, but I cannot prove this.

For $k > 4$, one cannot expect to express $F_k^{(j)}$ (even up to lower degree invariants) in terms of ∇, as the dimension of the space of Vassiliev invariants increases rapidly. Indeed, ∇_6 is not contained in F, as shown in [26] and [48]. That is, there are two distinct knots K_1 and K_2 with $F(K_1) = F(K_2)$ but $\nabla_6(K_1) \neq \nabla_6(K_2)$. As observed by Kanenobu, for the higher ∇_k the same property then follows by taking the connected sum of the $K_{1,2}$ with trefoils.

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