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FOLNER SETS OF ALTERNATE DIRECTED GROUPS

by Jérémie BRIEUSSEL

Abstract. — An explicit family of Folner sets is constructed for some directed groups acting on a rooted tree of sublogarithmic valency by alternate permutations. In the case of bounded valency, these groups were known to be amenable by probabilistic methods. The present construction provides a new and independent proof of amenability, using neither random walks, nor word length.

Résumé. — On construit une famille explicite d'ensembles de Folner pour certains groupes dirigés agissant sur des arbres enracinés à valence sous-logarithmique par des permutations alternées. Dans le cas d'arbres à valence bornée, la moyennabilité de ces groupes avait déjà été prouvée au moyen de techniques probabilistes. La construction présentée ici fournit une nouvelle preuve, n'utilisant ni marches aléatoires, ni longueur des mots.

1. Introduction

By a criterion of Folner [9], amenable groups are those that admit finite subsets with arbitrary small boundaries relatively to their cardinality. A sequence of such subsets, called a Folner sequence, is easily described for abelian groups, and well-understood for some classes of solvable groups ([17], [8]). Many non-solvable amenable groups are directed groups acting on rooted trees. This family of groups gathers many examples with "exotic" properties, such as infinite torsion groups of intermediate growth constructed by Aleshin ([1], [10]) or groups with non-uniform exponential growth by Wilson [19].

Their amenability in the case of bounded valency was shown in [6] by use of Kesten's probabilistic criterion [14]. The strategy, introduced by Bartholdi and Virag in [4], is to show that a self-similar random walk on a Cayley graph diffuses slowly, in the sense that its return probability does

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not decay exponentially, or that its entropy is sublinear ([13]). The same method permits to show that automata groups are amenable when their activity is bounded [3] or linear [2]. Though it ensures their existence, such a probabilistic proof does not exhibit Folner sets.

For the groups of [1] and [10], subexponential growth easily implies the existence of a subsequence of the family of balls (for a word length) which is a Folner sequence, but it is not known if the whole sequence of balls is Folner and the subsequence (even though it has density 1) is not explicit. Even for groups of polynomial growth, it is not elementary to show that balls form a Folner sequence, a result due to Pansu [16], using technics from Gromov [12].

The object of the present article is to exhibit explicit Folner sets for some groups with a property denoted $\mathcal{DP}$, satisfied in particular by directed groups acting on a rooted tree by alternate permutations. A group Γ with property $\mathcal{DP}$ is defined (see section 4) by two subgroups A finite and H finitely generated, together with an action on a rooted tree with valency sequence $(d_k)_{k\in\mathbb{N}}$. The main result is:

Theorem 1.1. — *Let Γ have property $\mathcal{DP}$ with H amenable and $\frac{d_k}{\log k} \to 0$, then the group Γ is amenable.*

In particular, the direct description of Folner sets provides a new proof, using neither random walks nor word length, that directed groups acting on a rooted tree of bounded valency are amenable ([6]). It also provides many new examples of amenable directed groups acting on a tree of unbounded sublogarithmic valency. Moreover it permits to reprove amenability of automata groups with bounded activity by methods different from [3].

The article is structured as follows. Rooted trees and their automorphism groups are described in section 2. Section 3 is devoted to the construction of explicit Folner sets for the archetypal example of the alternate mother group G_d acting on a regular rooted tree of valency $d \geqslant 5$. This example, treated first for simplicity of notations, is generalized to groups with property $\mathcal{DP}$ in section 4. Finally, section 5 is devoted to the construction of groups satisfying $\mathcal{DP}$, including the saturated alternate directed groups, and some groups acting on trees with unbounded valency.

2. Rooted trees and their groups of automorphisms

Let S_d denote the group of permutations of the set $\{1,\dots,d\}$ with d elements, and $\mathcal{A}_d = \mathcal{A}_{\{1,\dots,d\}}$ denote the subgroup of alternate permutations.

Given a sequence $\bar{d} = (d_k)_{k\geqslant 0}$ of integers $\geqslant 2$, the spherically homogeneous rooted tree $T_{\bar{d}}$ is the graph with vertex set $\{t_0t_1\cdots t_k \mid t_i \in$

$\{1,\ldots,d_i\}, k \geqslant -1\}$, including the empty sequence $\emptyset$, called the root, corresponding to $k=-1$, and edge set $\{(t_0\cdots t_k, t_0\cdots t_k t_{k+1})\}$. The vertex set restricted to a fixed k is called the $(k+1)$st level of the tree. It is the direct product $\{1,\ldots,d_0\}\times\cdots\times\{1,\ldots,d_k\}$. When the sequence $\bar{d}$ is constant equal to d, the tree is called d-regular, denoted T_d.

The group of automorphisms $\mathrm{Aut}(T_{\bar{d}})$ of the rooted tree $T_{\bar{d}}$ is the group of graph automorphisms that fix the root $\emptyset$. It satisfies a canonical isomorphism:

$$\varphi\colon\ \mathrm{Aut}(T_{\bar{d}}) \xrightarrow{\simeq} \mathrm{Aut}(T_{s\bar{d}}) \wr S_{d_0}, \tag{2.1}$$

where $s\bar{d}=(d_k)_{k\geqslant 1}$ is the shifted sequence obtained by deleting the first entry, and $G\wr S_d = (G\times\cdots\times G)\rtimes S_d$ is the semi-direct product where S_d acts by permuting factors, called wreath product. Since φ is canonical, we identify g and $\varphi(g)$ and write $g=(g_1,\ldots,g_{d_0})\sigma=(g_{t_0})\sigma$. The product rule is $gg'=(g_1g'_{\sigma(1)},\ldots,g_{d_0}g'_{\sigma(d_0)})\sigma\sigma'$, where g is applied before g'.

By iterating the wreath product isomorphism (2.1), a family of canonical isomorphisms is obtained:

$$\mathrm{Aut}(T_{\bar{d}}) \simeq \mathrm{Aut}(T_{s^k\bar{d}}) \wr S_{d_{k-1}} \wr\cdots\wr S_{d_0}. \tag{2.2}$$

Identifications are denoted $g=(g_{t_0\cdots t_k})(\sigma_{t_0\cdots t_{k-1}})\cdots(\sigma_{t_0})\sigma$, where $(\sigma_{t_0\cdots t_j})$ is a sequence of permutations in S_{d_j} indexed by the $(j+1)$st level of the tree and $(g_{t_0\cdots t_k})$ is a sequence of automorphisms of $T_{\sigma^k\bar{d}}$ indexed by level $k+1$. The automorphism g is determined by the whole sequence of permutations $(\sigma_v)_{v\in T_{\bar{d}}}$, called its portrait.

The automorphism g is said to be alternate if all the permutations σ_v of its portrait are alternate permutations. Denote $\mathrm{Aut}^{\mathrm{alt}}(T_{\bar{d}})$ the group of alternate automorphisms of $T_{\bar{d}}$. It also satisfies canonical isomorphisms:

$$\mathrm{Aut}^{\mathrm{alt}}(T_{\bar{d}}) \simeq \mathrm{Aut}^{\mathrm{alt}}(T_{s^k\bar{d}}) \wr \mathcal{A}_{d_{k-1}} \wr\cdots\wr \mathcal{A}_{d_0}.$$

The neutral element of a group G is denoted e_G or e.

3. Folner sets of the alternate mother group

3.1. The alternate mother group

In the case of a d-regular rooted tree T_d, the canonical wreath product isomorphism of the group of alternate automorphisms has the form:

$$\varphi\colon\ \mathrm{Aut}^{\mathrm{alt}}(T_d) \xrightarrow{\simeq} \mathrm{Aut}^{\mathrm{alt}}(T_d)\wr\mathcal{A}_d. \tag{3.1}$$

It permits to define recursively some alternate automorphisms of T_d as follows.

- Given σ in $\mathcal{A}_d$, denote $A = \{a(\sigma) \mid \sigma \in \mathcal{A}_d\} \simeq \mathcal{A}_d$ with:
$$\varphi(a(\sigma)) = (e, \ldots, e)\sigma.$$
The elements $a = a(\sigma)$ of A are alternate automorphisms of T_d, called rooted automorphisms, because the portrait of $a(\sigma)$ is given by $\sigma_\emptyset = \sigma$ and $\sigma_v = e$ for $v \neq \emptyset$.
- Given $a_2, \ldots, a_d$ in $\mathcal{A}_d$ and ρ in $\mathrm{Fix}_{\mathcal{A}_d}(1) = \mathcal{A}_{\{2,\ldots,d\}} = \mathcal{A}_{d-1}$, the alternate automorphism $b = b(a_2, \ldots, a_d, \rho)$ satisfies under the wreath product isomorphism:
$$\varphi\big(b(a_2, \ldots, a_d, \rho)\big) = \big(b(a_2, \ldots, a_d, \rho), a_2, \ldots, a_d\big)\rho.$$
This defines recursively a tree automorphism $b = b(a_2, \ldots, a_d, \rho)$ with portrait the family of permutations $(\sigma_v)_{v \in T_d}$ given by $\sigma_{1\cdots 11} = \rho$, $\sigma_{1\cdots 1t} = a_t$ for $2 \leqslant t \leqslant d$ and $\sigma_v = e$ for the other vertices v.
Denote $B = \{b(a_2, \ldots, a_d, \rho) \mid a_2, \ldots, a_d \in \mathcal{A}_d, \rho \in \mathrm{Fix}_{\mathcal{A}_d}(1)\}$. The elements of B are called directed. The set B forms a finite subgroup of $\mathrm{Aut}(T_d)$. Indeed, the following is an isomorphism:
$$\begin{array}{rcl} B & \to & (\mathcal{A}_d \times \cdots \times \mathcal{A}_d) \rtimes \mathcal{A}_{\{2,\ldots,d\}} \\ b(a_2, \ldots, a_d, \rho) & \mapsto & (a_2, \ldots, a_d)\rho. \end{array} \quad (3.2)$$
- The alternate mother group G_d is the subgroup of alternate automorphisms of T_d generated by the sets A, B:
$$G_d = \langle A, B \rangle < \mathrm{Aut}^{\mathrm{alt}}(T_d).$$

By construction, the group G_d is an automata group. It is essentially the mother group of degree 0 (see [3], [2]), but the permutations involved are alternate. Since $\mathcal{A}_d$ is simple hence perfect for $d \geqslant 5$, the group G_d satisfies the:

Proposition 3.1. — *If $d \geqslant 5$, the canonical isomorphism (3.1) induces an isomorphism:*
$$\varphi\colon G_d \xrightarrow{\simeq} G_d \wr \mathcal{A}_d.$$

This isomorphism will also be considered canonical $G_d \simeq G_d \wr \mathcal{A}_d$, and the elements g and $\varphi(g)$ will be identified in the remainder of this section. The proposition follows from the:

Fact 3.2. — *Let $d \geqslant 5$, then for any generator $a = a(\sigma) \in A$ and $b = b(a_2, \ldots, a_d, \rho) \in B$, the elements $(a, e, \ldots, e)$ and $(b, e, \ldots, e)$ belong to $\varphi(G_d)$.*

Recall the conjugacy notation $g^a = aga^{-1}$, and observe that for $g = (g_1, \ldots, g_d)\sigma$ and a in A, one has $g^a = (g_{a(1)}, \ldots, g_{a(d)})\sigma^a$.

Proof of Fact 3.2. — Take τ in $\mathcal{A}_d$ such that $\tau(1)=1$ and $\tau^{-1}(2)=3$ and observe the commutator relations:

$$\begin{aligned}\varphi(b) &= \varphi(b(\alpha_2, e_A, \dots, e_A, e_A)) = (b, \alpha_2, e_G, e_G, \dots, e_G)e_A,\\ \varphi(b'^{\tau}) &= \varphi(b(\alpha'_2, e_A, \dots, e_A, e_A)^{\tau}) = (b', e_G, \alpha'_2, e_G, \dots, e_G)e_A,\\ \varphi([b, b'^{\tau}]) &= ([b, b'], e_G, e_G, e_G, \dots, e_G).\end{aligned}$$

As $[b(\alpha_2, e_A, \dots, e_A, e_A), b(\alpha'_2, e_A, \dots, e_A, e_A)] = b([\alpha_2, \alpha'_2], e_A, \dots, e_A, e_A)$ and as the group $A \simeq \mathcal{A}_d$ is perfect (because it is simple), any element a_2 in $A \simeq \mathcal{A}_d$ is a product of commutators. This shows that $\varphi(G_d)$ contains $(b_2, e_G, \dots, e_G)$ for any $b_2 = b(a_2, e_A, \dots, e_A, e_A)$ with a_2 in $\mathcal{A}_d$. Moreover for any $b_\emptyset = b(e_A, \dots, e_A, \rho)$ with ρ in $\mathrm{Fix}_A(1) \simeq \mathcal{A}_{d-1}$, the group $\varphi(G_d)$ contains $\varphi(b_\emptyset a(\rho^{-1})) = (b_\emptyset, e_G, \dots, e_G)$.

Now the elements $b_2 = b(a_2, e_A, \dots, e_A, e_A)$ and $b_\emptyset = b(e_A, \dots, e_A, \rho)$ generate B by isomorphism (3.2), because ρ in $\mathcal{A}_{\{2,\dots,d\}}$ and $(a_2, e_A, \dots, e_A)$ for a_2 in $\mathcal{A}_d$ generate the finite group $(\mathcal{A}_d \times \dots \times \mathcal{A}_d) \rtimes \mathcal{A}_{\{2,\dots,d\}}$. Thus $\varphi(G_d)$ contains $(b, e_G, \dots, e_G)$ for any b in B.

Finally given a_2 in A, for $b_2 = b(a_2, e_A, \dots, e_A, e_A) = (b_2, a_2, e_G, \dots, e_G)$, the element $(b_2^{-1}, e_G, \dots, e_G)$ belongs to $\varphi(G_d)$ by the above. So do $(b_2^{-1}, e_G, \dots, e_G)\varphi(b_2) = (e_G, a_2, e_G, \dots, e_G)$ and $(e_G, a_2, e_G, \dots, e_G)^{\tau} = (a_2, e_G, \dots, e_G)$ for τ in $\varphi(A) \simeq A \simeq \mathcal{A}_d$ such that $\tau^{-1}(2) = 1$. □

Proof of Proposition 3.1. — By definition of the generators of G_d, the morphism φ is an embedding into the wreath product $G_d \wr \mathcal{A}_d$. The key point is that this embedding is surjective. Clearly $\varphi(A) \simeq A \simeq \mathcal{A}_d$ is the set of rooted automorphisms. Moreover, Fact 3.2 shows that $G_d \times \{e\} \times \dots \times \{e\}$ is in $\varphi(G_d)$. As $\mathcal{A}_d$ acts transitively on $\{1, \dots, d\}$, conjugation shows that $\{e\} \times \dots \times G_d \times \dots \times \{e\}$ also belongs to $\varphi(G_d)$ for any position of the non-trivial factor. Then $G_d \times \dots \times G_d$ belongs to $\varphi(G_d)$ by product. This proves the wreath product isomorphism. □

3.2. Definition of Folner sets

For a group Γ with finite generating set S, the boundary of a subset $L \subset \Gamma$ is defined as:

$$\partial_S L = \left\{\gamma \in L \mid \exists s \in S, \gamma s \notin L\right\}.$$

The interior of L is the set $\mathrm{Int}_S(L) = L \setminus \partial_S L$.

A sequence L_k of subsets of Γ is a Folner sequence if $\frac{|\partial L_k|}{|L_k|} \to 0$. By [9], a finitely generated group Γ is amenable if and only if it admits a Folner sequence for some (equivalently for any) finite generating set S. For the remainder of this section, the set $S = A \cup B$ is considered the canonical generating set of G_d and the notations ∂L and $\mathrm{Int}(L)$ stand for $\partial_{A\cup B} L$ and $\mathrm{Int}_{A\cup B}$ respectively.

Let us define a sequence of subsets of G_d as follows:

$$L_0 = \left\{ g \in G_d \mid \exists \beta \in B, \alpha_2, \dots, \alpha_d, \sigma \in A, g = (\beta, \alpha_2, \dots, \alpha_d)\sigma \right\}.$$

By induction on k, define:

$$L_{k+1} = \left\{ g \in G_d \,\middle|\, \begin{array}{r} g = (g_1, \dots, g_d)\sigma, \text{ such that } \sigma \in A, \forall t \in \{1, \dots, d\}, g_t \in L_k \\ \text{and } \exists T \in \{1, \dots, d\}, g_T \in \mathrm{Int}(L_k) \end{array} \right\}.$$

By Proposition 3.1, the sets L_k are included in G_d for $d \geqslant 5$, and not just in the automorphism group $\mathrm{Aut}(T_d)$.

Theorem 3.3. — *For $d \geqslant 5$, the sets L_k form a Folner sequence for G_d. In particular, the group G_d is amenable.*

The group G_d was known to be amenable by [6] (use of Kesten criterion on return probability) or [3] (triviality of the Poisson boundary). However, these proofs, based on contraction in the wreath product of word length for some random walks, did not provide explicit Folner sets. The following proof uses neither random walks, nor word length.

Remark 3.4. — Estimation on the rate of convergence of $|\partial L_k|/|L_k|$ to zero and on the cardinality of L_k will show that the Folner function $\mathrm{Fol}(n) = \min\{|L|/n|\partial L| \leqslant |L|\}$ is bounded above by a function $\exp_C \times(\exp_C(n^{d-1+\varepsilon}))$ for some constant C and any positive ε where $\exp_C(x) = C^x$. However, the return probability of a symmetric simple random walk on G_d is bounded below by $\exp(-n^{\beta_d})$ for $\beta_d = \log d / \log \frac{d^2}{d-1}$ by Theorem 6.1 in [7]. Combined with Corollary V.2 in [11], this shows that the function $\mathrm{Fol}(n)$ cannot be bounded below by $\exp(n^\alpha)$ for $\alpha = 2\beta_d/1 - \beta_d$. Thus the sequence L_k exhibited here is far from optimal as a Folner sequence.

3.3. Proof of Theorem 3.3

Observe that for any a in A and $g = (\beta, \alpha_2, \dots, \alpha_d)\sigma$ in L_0, the element $ga = (\beta, \alpha_2, \dots, \alpha_d)\sigma a$ still belongs to L_0. Moreover, for any $b = b(a_2, \dots, a_d, \rho) = (b, a_2, \dots, a_d)\rho$ in B, one has:

$$gb = \begin{cases} (\beta b, \alpha_2 a_{\sigma(2)}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) = 1, \\ (\beta a_{\sigma(1)}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_{\sigma^{-1}(1)} b, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) \neq 1. \end{cases}$$

As the sets A and B are finite groups, this shows equivalence of (1), (2) and (3) in the:

Fact 3.5. — *The following are equivalent:*

1. *g belongs to* $\mathrm{Int}(L_0)$,
2. *$gb \in L_0$ for all $b \in B$,*
3. *$\sigma^{-1}(1) = 1$,*
4. *$gb \in \mathrm{Int}(L_0)$ for all $b \in B$.*

In particular, $\frac{|\mathrm{Int}(L_0)|}{|L_0|} = \frac{1}{d}$, *hence* $\delta_0 = \frac{|\partial L_0|}{|L_0|} = 1 - \frac{1}{d}$.

Proof. — Point (4) is equivalent to (3) due to the fixed point assumption $\rho(1) = 1$ in the definition of B, which guarantees that $(\sigma\rho)^{-1}(1) = (\rho^{-1}\sigma^{-1})(1) = \sigma^{-1}(\rho^{-1}(1)) = 1$ when $\sigma^{-1}(1) = 1$.

The evaluation of δ_0 is done by counting $|L_0| = |B||A|^d$ as g is described by $\beta, \alpha_2, \ldots, \alpha_d, \sigma$, and condition $\sigma^{-1}(1) = 1$ occurs with probability $\frac{1}{d}$. □

Lemma 3.6. — *Let $g \in L_k$, the following are equivalent:*

1. *g belongs to* $\mathrm{Int}(L_k)$,
2. *$gb \in L_k$ for all $b \in B$,*
3. *$\sigma^{-1}(1) \in I(g) = \{T \mid g_T \in \mathrm{Int}(L_{k-1})\}$,*
4. *$gb \in \mathrm{Int}(L_k)$ for all $b \in B$.*

Proof of Lemma 3.6. — The case $k = 0$ is treated by Fact 3.5 with convention that $I(g) = \{1\}$ if $g \in L_0$. Assume by induction that the result is true for $k - 1$, and prove it for k.

Again $ga = (g_1, \ldots, g_d)\sigma a$ belongs to L_k for any value of a in A, g in L_k. Moreover:

$$gb = \big(g_1 a_{\sigma(1)}, \ldots, g_{\sigma^{-1}(1)} b, \ldots, g_d a_{\sigma(d)}\big)\sigma\rho.$$

Suppose (3) holds true, that is $g_{\sigma^{-1}(1)} \in \mathrm{Int}(L_{k-1})$, then as (1) implies (4) for $k - 1$, the element $g_{\sigma^{-1}(1)} b$ belongs to $\mathrm{Int}(L_{k-1})$ for any b in B, so that gb belongs to L_k for any b in B, proving (2). Then (1) follows because ga also belongs to L_k for a in A, hence g is an interior point of L_k.

Suppose (3) does not hold, so $g_{\sigma^{-1}(1)} \in \partial L_{k-1}$. By equivalence of (1) and (2) for $k - 1$, there exists b in B such that $g_{\sigma^{-1}(1)} b \notin L_{k-1}$, so that gb is not in L_k, disclaiming (1) and (2) for g. This proves equivalence of (1), (2) and (3) for k.

Now gb belongs to $\mathrm{Int}(L_k)$ if and only if $(\sigma\rho)^{-1}(1) \in I(g)$ by equivalence of (1) and (3). But $(\sigma\rho)^{-1}(1) = \sigma^{-1}(\rho^{-1}(1)) = \sigma^{-1}(1)$ because $\rho(1) = 1$. So (3) implies (4). Obviously, (4) implies (2), closing step k of induction. □

There remains to evaluate the sizes of the interior and boundary of L_k. Set:

$$\delta_k = \frac{|\partial L_k|}{|L_k|}, \quad 1-\delta_k = \frac{|\operatorname{Int}(L_k)|}{|L_k|}.$$

Lemma 3.7. — *The sequence (δ_k) satisfies:*

$$1-\delta_{k+1} = \frac{1-\delta_k}{1-\delta_k^d}.$$

Proof of Lemma 3.7. — Given a subset $I \subset \{1,\dots,d\}$, denote:

$$J_I = \big\{g = (g_1,\dots,g_d)\sigma \mid \forall T \in I, g_T \in \operatorname{Int}(L_k) \text{ and } \forall t \notin I, g_t \in \partial L_k\big\}.$$

By definition, L_{k+1} is the disjoint union $L_{k+1} = \sqcup_{|I|\geqslant 1} J_I$.

For $i = |I|$, the size of J_I and its intersection with $\operatorname{Int}(L_{k+1})$ are evaluated as:

$$|J_I| = |\mathcal{A}_d||\operatorname{Int}(L_k)|^i|\partial L_k|^{d-i} = |\mathcal{A}_d||L_k|^d(1-\delta_k)^i\delta_k^{d-i},$$

$$|J_I \cap \operatorname{Int}(L_{k+1})| = \frac{|I|}{d}|\mathcal{A}_d||\operatorname{Int}(L_k)|^i|\partial L_k|^{d-i} = \frac{i}{d}|J_I|,$$

where the factor $\frac{i}{d}$ comes from (3) of Lemma 3.6. Denote C_d^i the number of subsets of size i in $\{1,\dots,d\}$, and use the mean of binomial distribution to get:

$$|\operatorname{Int}(L_{k+1})| = \sum_{i=1}^{d} C_d^i(1-\delta_k)^i\delta_k^{d-i}\frac{i}{d}|L_k|^d|\mathcal{A}_d| = (1-\delta_k)|L_k|^d|\mathcal{A}_d|,$$

$$|L_{k+1}| = \sum_{i=1}^{d} C_d^i(1-\delta_k)^i\delta_k^{d-i}|L_k|^d|\mathcal{A}_d| = (1-\delta_k^d)|L_k|^d|\mathcal{A}_d|.$$

This shows that:

$$1-\delta_{k+1} = \frac{|\operatorname{Int}(L_{k+1})|}{|L_{k+1}|} = \frac{1-\delta_k}{1-\delta_k^d}.$$

□

Proof of Theorem 3.3. — As $\delta_k > 0$, Lemma 3.7 implies $1-\delta_{k+1} > 1-\delta_k$, so the sequence (δ_k) is decreasing, tending to a limit δ satisfying $1-\delta = \frac{1-\delta}{1-\delta^d}$, hence δ is 0 (or 1, ruled out by $\delta_0 < 1$). □

More precisely, Lemma 3.7 implies that for any $\eta < \frac{1}{d-1}$, one has $\delta_k = O(k^{-\eta})$, as shown below in Lemma 4.13. On the other hand:

$$|L_k| = |B|^{d^k}|A|^{(d-1)d^k+(d^k+\cdots+d+1)} \geqslant 2^{2^k}.$$

This provides the estimate on the Folner function in remark 3.4.

Remark 3.8. — Lemma 3.6 provides a complete combinatorial description of L_k. An element g of G_d has the form $g = (g_{t_0\cdots t_k})(\sigma_{t_0\cdots t_{k-1}})\cdots(\sigma_{t_0})\sigma$ in the kth iteration of the wreath product. Such an element g belongs to L_k if and only if it satisfies the three following conditions:

(1) $\forall t_0\cdots t_{k-1}$, the element $g_{t_0\cdots t_{k-1}1}$ is in B and $g_{t_0\cdots t_{k-1}2},\dots,$ $g_{t_0\cdots\cdots t_{k-1}d}$ are in A,
(2) $\forall t_0\cdots t_{k-2}$, the set $I(t_0\cdots t_{k-2}) = \{T_{k-1} \mid \sigma^{-1}_{t_0\cdots t_{k-2}T_{k-1}}(1) = 1\}$ is non-empty.
(3) $\forall 3 \leqslant l \leqslant k+1, \forall t_0\cdots t_{k-l}$, the set

$$I(t_0\cdots t_{k-l}) = \{T_{k-l+1} \mid \sigma^{-1}_{t_1\cdots t_{k-l}T_{k-l+1}}(1) \in I(t_1\cdots t_{k-l}T_{k-l+1})\},$$

defined by induction on l, is non-empty (for $l = k+1$, consider $I(\emptyset)$ where $\emptyset$ is the root vertex of T_d).

The element g belongs to $\mathrm{Int}(L_k)$ if and only if it satisfies (1), (2), (3) and moreover:

(4) $\sigma^{-1}(1) \in I(\emptyset) = \{T \mid \sigma_T \in I(T)\}$.

Note that condition (2) is a specific case of condition (3) where $I(t_0\cdots t_{k-1}) = \{1\}$ for all $t_0\cdots t_k$. As an interpretation, say a vertex $v = t_0\cdots t_l$ with $l \leqslant k-1$ is open if $\sigma_v^{-1}(1) \in I(v)$. Conditions (1), (2), (3) ensure that g belongs to L_k if and only if each vertex v has at least one neighbour of next level vT which is open. Condition (4) ensures that g is in the interior $\mathrm{Int}(L_k)$ if and only if the root itself is open.

4. Generalization

4.1. Property $\mathcal{DP}$

Theorem 3.3 can be generalized to the following wider setting.

DEFINITION 4.1. — *A sequence of groups is said to have property $\mathcal{DP}$ if it satisfies the two following conditions for all i in $\mathbb{N}$:*

(1) *the group Γ_i contains two subgroups A_i and H_i such that:*
  (a) *the set $A_i \cup H_i$ generates the group Γ_i,*
  (b) *the group A_i is finite, acting transitively on a finite set $\{1,\dots,d_i\}$ of size $d_i \geqslant 2$,*
  (c) *the group H_i is finitely generated,*
(2) *there is an isomorphism:*

$$\varphi_i\colon \Gamma_i \longrightarrow \Gamma_{i+1} \wr A_i = (\Gamma_{i+1}\times\cdots\times\Gamma_{i+1}) \rtimes A_i,$$

with d_i factors in the direct product, on which A_i is acting by permutation of coordinates, according to its transitive action on $\{1,\dots,d_i\}$. Moreover, this isomorphism φ_i satisfies:

(a) $\forall s \in A_i, \varphi_i(s) = (e_{\Gamma_{i+1}},\dots,e_{\Gamma_{i+1}})s$,

(b) $\forall h_i \in H_i, \exists h_{i+1} \in H_{i+1}, \exists a_2,\dots,a_{d_0} \in A_{i+1}, \exists \rho \in A_i$, *with* $\rho(1) = 1$ *and:*

$$\varphi_i(h_i) = (h_{i+1}, a_2, \dots, a_{d_i})\rho,$$

where the groups A_{i+1} and H_{i+1} are the subgroups of Γ_{i+1} satisfying condition (1).

A group Γ is said to have property $\mathcal{DP}$ if there exists a sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ with property $\mathcal{DP}$ such that $\Gamma \simeq \Gamma_0$.

Groups with property $\mathcal{DP}$ are related to the groups of non-uniform growth constructed by Wilson (see [19],[18],[6]). In particular, if all the groups Γ_i of a sequence with property $\mathcal{DP}$ are generated by a finite number (independent of i) of involutions, and if all the groups A_i involved are alternate groups $\mathcal{A}_{d_i}$ acting on sets of size $d_i \geqslant 29$, then they have non-uniform growth by [18]. This is the case of the examples in proposition 5.3 below.

Fact 4.2. — *If Γ_0 belongs to $\mathcal{DP}$, there exists a sequence $\bar{d} = (d_i)_i$ of integers $d_i \geqslant 2$, and the group Γ_0 is acting by automorphisms on the spherically homogeneous rooted tree $T_{\bar{d}}$. This action is transitive on each level.*

Note that this action on the tree is not necessarily faithful. For instance, the subgroup F of the group $\Gamma = \Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, F)$ of section 2.4 of [7] has a trivial action on the tree $T_{\bar{d}}$, even though Γ has property $\mathcal{DP}$, for the sequence $\Gamma_i = \Gamma(\mathcal{A}_{d_i}, A_{s^i\bar{d}}, F)$.

Proof. — Combining the isomorphisms of Definition 4.1, there is an isomorphism $\Gamma_0 \simeq \Gamma_{i+1} \wr A_i \wr \cdots \wr A_0$. As A_i is acting transitively on $\{1,\dots,d_i\}$, the group $A_i \wr \cdots \wr A_0$ is acting transitively on $\{1,\dots,d_0\} \times \cdots \times \{1,\dots,d_i\}$, which is the $i+1$st level of $T_{\bar{d}}$. Taking the limit with i, this provides the action on the tree $T_{\bar{d}}$. □

Consider a group Γ_0 with property $\mathcal{DP}$ and take notations of Definition 4.1. Let B_0 be a fixed finite generating set of H_0. Define inductively the sequence B_i of subsets of H_i by condition (2)(b). For any $b_i \in B_i$, set b_{i+1} to be the unique element in H_{i+1} such that $\varphi_i(b_i) = (b_{i+1}, a_2, \dots, a_{d_i})\rho$, and $B_{i+1} = \{b_{i+1} \mid b_i \in B_i\}$. By construction, B_i is a subset of H_i of size

$\leqslant |B_0|$. It is not true in general that B_i generates H_i for all $i \in \mathbb{N}$, however, we have:

FACT 4.3. — *Let Γ_0 have property $\mathcal{DP}$ and B_i be as above. Then the conditions of Definition 4.1 are fulfilled if H_i is replaced by the subgroup $\langle B_i \rangle$ of Γ_i.*

Proof. — It is sufficient to check conditions (1) and (2) of Definition 4.1 for all i in $\mathbb{N}$. For $i = 0$, (1) is true since $H_0 = \langle B_0 \rangle$, and (2) is true by definition of B_1. Then:

$$\Gamma_1 \wr A_0 \simeq \varphi_0(\Gamma_0) \subset \langle B_1 \cup A_1 \rangle \wr A_0.$$

The inclusion is forced to be an equality since A_1, B_1 are included in Γ_1, thus $A_1 \cup B_1$ generates Γ_1. This shows that H_1 can be replaced by $\langle B_1 \rangle$. The fact follows by induction. □

This shows that up to replacing the groups H_i by the groups $\langle B_i \rangle$, which does not affect the groups Γ_i, we may and shall assume that B_i is a canonical generating set for H_i.

FACT 4.4. — *The group H_0 is amenable if and only if the groups H_i are amenable for all i.*

Proof. — By (2)(b), the restriction of φ_0 to H_0 provides an embedding:

$$\varphi_0\big|_{H_0} \colon H_0 \hookrightarrow H_1 \times \big(A_1 \wr \mathrm{Fix}_{A_0}(1)\big).$$

As the second factor is a finite group, amenability of H_1 implies that of H_0.

Conversely assume that H_0 is amenable. By (2)(b), any relation between the generators in B_0 implies a relation between the corresponding generators in B_1 of H_1. Thus H_1 is a quotient of H_0, hence is amenable.

The same proof shows that amenability of H_{i+1} is equivalent to that of H_i. □

Question 4.5. — If a group $\Gamma_0 = \langle A_0 \cup H_0 \rangle$ has property $\mathcal{DP}$ with H_0 amenable, is the group Γ_0 amenable?

The following theorem provides a partial answer, with a condition on the sequence of integers $\bar{d} = (d_i)_i$.

THEOREM 4.6. — *Let Γ_0 have property $\mathcal{DP}$ with H_0 amenable and $\bar{d}$ growing sufficiently slowly (for instance $\frac{d_k}{\log k} \to 0$), then Γ_0 is amenable.*

This theorem generalizes Theorem 3.3. The proof is similar, though slightly more technical.

4.2. Proof of Theorem 4.6

Given $\Gamma_0 = \langle A_0 \cup B_0 \rangle$ with property $\mathcal{DP}$, consider the associated sequence of finitely generated groups $\Gamma_K = \langle A_K \cup B_K \rangle$, where B_K is the canonical generating set of the group H_K. The notions of interior and boundary used below refer to these generating sets.

To ease notations, write g instead of $\varphi_K(g)$. For $\Omega \subset H_{K+1}$, set:

$$L_0^K(\Omega) = \big\{ g \in \Gamma_K \mid \exists h \in \Omega, \alpha_2, \dots, \alpha_{d_K} \in A_{K+1}, \\ \sigma \in A_K, g = (h, \alpha_2, \dots, \alpha_{d_K})\sigma \big\},$$

$$\iota L_0^K(\Omega) = \big\{ g \in L_0^K(\Omega) \mid \sigma^{-1}(1) = 1 \big\},$$

and by induction for $1 \leqslant k \leqslant K$, set:

$$L_k^K(\Omega) = \big\{ g = (g_1, \dots, g_{d_{K-k}})\sigma \in \Gamma_{K-k} \mid \forall t, g_t \in L_{k-1}^K(\Omega), \\ \exists T, g_T \in \iota L_{k_1}^K(\Omega) \big\},$$

$$\iota L_k^K(\Omega) = \big\{ g \in L_k^K(\Omega) \mid g_{\sigma^{-1}(1)} \in \iota L_{k-1}^K(\Omega) \big\}.$$

The sets $\iota L_k^K(\Omega)$ should be considered as "combinatorial interiors" of $L_k^K(\Omega)$. They satisfy a combinatorial description as Remark 3.8, but slightly differ from the actual interior of $L_k^K(\Omega)$, unless the set Ω has empty boundary (see Remark 4.10 below). Fact 3.5 generalizes as:

Fact 4.7. — *The three following are equivalent:*

1. $g \in \mathrm{Int}(L_0^K(\Omega))$,
2. $g b_K \in L_0^K(\Omega)$ *for all* $b_K \in B_K$,
3. $\sigma^{-1}(1) = 1$ *and* $h \in \mathrm{Int}(\Omega) \subset \Omega \subset H_{K+1}$.

   Moreover they also imply:

4. $g b_K \in \iota L_0^K(\Omega)$ *for all* $b_K \in B_K$.

In particular, $\frac{|\mathrm{Int}(L_0^K(\Omega))|}{|L_0^K(\Omega)|} = \frac{|\mathrm{Int}(\Omega)|}{d_K|\Omega|}$, *and* $\delta_0^K(\Omega) = \frac{|\partial L_0^K(\Omega)|}{|L_0^K(\Omega)|} = 1 - \frac{|\mathrm{Int}(\Omega)|}{d_K|\Omega|}$.

Proof. — Let $g = (h, \alpha_2, \dots, \alpha_{d_K})\sigma$ belong to $L_0^K(\Omega)$. By (2)(a) of Definition 4.1 for A_K, the element $g a_K$ still belongs to $L_0^K(\Omega)$ for a_K in A_K. This proves equivalence of (1) and (2).

Now take $b_K = (b_{K+1}, a_2, \dots, a_{d_K})\rho$ in B_K, then:

$$g b_K = \begin{cases} (h b_{K+1}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) = 1, \\ (h a_{\sigma(1)}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_{\sigma^{-1}(1)} b_{K+1}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) \neq 1. \end{cases}$$

This shows that $g b_K$ belongs to $L_0^K(\Omega)$ for all b_K if and only if $\sigma^{-1}(1) = 1$ and h belongs to $\mathrm{Int}(\Omega)$, *i.e.* equivalence of (2) and (3).

This implies (4) because then $(\sigma\rho)^{-1}(1) = 1$. Computing the sizes follows from (3). $\square$

Notation 4.8. — Let $g = (g_1, \dots, g_{d_i})\sigma = (g_{t_i})\sigma$ in Γ_i, with σ in A_i, g_{t_i} in Γ_{i+1} for $t_i \in \{1, \dots, d_i\}$ by identification of g with $\varphi_i(g)$. More generally, identify $g_{t_i \cdots t_j}$ with $\varphi_{j+1}(g_{t_i \cdots t_j})$ for $i \leqslant j \leqslant K$ and denote:

$$g = (g_{t_i \cdots t_K})(\sigma_{t_i \cdots t_{K-1}}) \cdots (\sigma_{t_i})\sigma,$$

where $\sigma_{t_i \cdots t_j}$ belongs to A_{j+1} and $g_{t_i \cdots t_K}$ to Γ_{K+1}. Set $\tau_i = \sigma^{-1}(1) \in \{1, \dots, d_i\}$, and by induction $\tau_{j+1} = (\sigma_{\tau_i \cdots \tau_j})^{-1}(1) \in \{1, \dots, d_{j+1}\}$, which guarantees $g(\tau_i \tau_{i+1} \cdots \tau_j) = 11 \cdots 1$ for the action on the tree of fact 4.2.

The following generalizes Lemma 3.6.

Lemma 4.9. — *For $0 \leqslant k \leqslant K$, the three following are equivalent:*

1. $g \in \mathrm{Int}(L_k^K(\Omega))$,
2. $gb_{K-k} \in L_k^K(\Omega)$ *for all* $b_{K-k} \in B_{K-k}$,
3. $g \in \iota L_k^K(\Omega)$ *(i.e.* $\sigma^{-1}(1) \in I(g) = \{T \mid g_T \in \iota L_{k-1}^K(\Omega)\}$*) and* $g_{\tau_{K-k} \cdots \tau_K} \in \mathrm{Int}(\Omega)$.

   Moreover, they also imply:
4. $gb_{K-k} \in \iota L_k^K(\Omega)$ *for all* $b_{K-k} \in B_{K-k}$.

Observe that if $g \in \iota L_k^K(\Omega)$, then $g_{\tau_{K-k} \cdots \tau_K} \in \Omega$, by definitions of $\iota L_k^K(\Omega)$ and $\tau_{K-k} \cdots \tau_K$.

Proof. — Let $g = (g_1, \dots, g_{d_{K-k}})\sigma$ belong to $L_k^K(\Omega)$. For a in A_{K-k}, ga still belongs to $L_k^K(\Omega)$ (no condition on σ). Thus (1) is equivalent to (2). To prove equivalence with (3) and implication of (4), proceed by induction on $0 \leqslant k \leqslant K$. The case $k = 0$ was treated as fact 4.7 (where $h = g_1 = g_{\sigma^{-1}(1)} = g_{\tau_K}$), now assume the lemma is known for $k - 1$.

For $b_{K-k} = \big(b_{K-k+1}, a_2, \dots, a_{d_{K-k}}\big)\rho$, one has:

$$gb_{K-k} = \big(g_1 a_{\sigma(1)}, \dots, g_{\sigma^{-1}(1)} b_{K-k+1}, \dots, g_{d_{K-k}} a_{\sigma(d_{K-k})}\big)\sigma\rho.$$

Assume (2) for g, then $g_{\sigma^{-1}(1)} b_{K-k+1} \in L_{k-1}^K(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$, which means (2) for $k - 1$ applied to $g_{\sigma^{-1}(1)}$. By induction hypothesis, $g_{\sigma^{-1}(1)}$ satisfies (3), which means that it belongs to $\iota L_{k-1}^K(\Omega)$, so $g \in \iota L_k^K(\Omega)$, and $g_{\sigma^{-1}(1)\tau_{K-k+1} \cdots \tau_K} = g_{\tau_{K-k}\tau_{K-k+1} \cdots \tau_K} \in \mathrm{Int}(\Omega)$, proving (3) for g.

Moreover, (2) applied to $g_{\sigma^{-1}(1)}$ implies, by induction, (4) that $g_{\sigma^{-1}(1)} b_{K-k+1} \in \iota L_{k-1}^K(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$. As $(\sigma\rho)^{-1}(1) = \sigma^{-1}(\rho^{-1}(1)) = \sigma^{-1}(1)$, this shows $gb_{K-k} \in \iota L_{K-k}^K(\Omega)$, which is (4) for g.

Conversely, assume (3) for g, then $g_{\sigma^{-1}(1)} \in \iota L_{k-1}^K(\Omega)$, and $g_{\tau_{K-k}\tau_{K-k+1} \cdots \tau_K} = g_{\sigma^{-1}(1)\tau_{K-k+1} \cdots \tau_K} \in \mathrm{Int}(\Omega)$, *i.e.* (3) for $g_{\sigma^{-1}(1)}$. As (3) implies (4) for $k-1$, one has $g_{\sigma^{-1}(1)}b_{K-k+1} \in \iota L^K_{k-1}(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$, so $gb_{K-k} \in L^K_k(\Omega)$ for all $b_{K-k} \in B_{K-k}$, which means (2) for g. □

Remark 4.10. — The combinatorial description of Remark 3.8 still applies to an element $g \in \Gamma_{K-k}$ of the form:

$$g = (g_{t_{K-k}\cdots t_K})(\sigma_{t_{K-k}\cdots t_{K-1}})\cdots(\sigma_{t_{K-k}})\sigma,$$

with $t_{K-k+l} \in \{1, \dots, d_{K-k+l}\}$, $\sigma_{t_{K-k}\cdots t_{K-k+l}} \in A_{K-k+l+1}$ and $g_{t_{K-k}\cdots t_K} \in \Gamma_{K+1}$. Such an element g belongs to $L^K_k(\Omega)$ if and only if it satisfies the three following conditions:

(1) $\forall t_{K-k}\cdots t_{K-1}$, the element $g_{t_{K-k}\cdots t_{K-1}1}$ is in $\Omega \subset H_{K+1}$ and the elements $g_{t_{K-k}\cdots t_{K-1}2}, \dots, g_{t_{K-k}\cdots t_{K-1}d_K}$ are in A_{K+1},

(2) $\forall t_{K-k}\cdots t_{K-2}$, the set:

$$\begin{aligned} &I(t_{K-k}\cdots t_{K-2}) \\ &\quad = \{T_{K-1} \in \{1, \dots, d_{K-1}\} \mid \sigma^{-1}_{t_{K-k}\cdots t_{K-2}T_{K-1}}(1) = 1\} \\ &\quad = \{T_{K-1} \in \{1, \dots, d_{K-1}\} \mid g_{t_{K-k}\cdots t_{K-2}T_{K-1}} \in \iota L^K_0(\Omega) \subset \Gamma_K\} \end{aligned}$$

is non-empty.

(3) $\forall 2 \leqslant l \leqslant k, \forall t_{K-k}\cdots t_{K-l}$, the following subset of $\{1, \dots, d_{K-l+1}\}$:

$$\begin{aligned} &I(t_{K-k}\cdots t_{K-l}) \\ &\quad = \left\{T_{K-l+1} \mid \sigma^{-1}_{t_{K-k}\cdots t_{K-l}T_{K-l+1}}(1) \in I(t_{K-k}\cdots t_{K-l}T_{k-l+1})\right\}, \\ &\quad = \left\{T_{K-l+1} \mid g_{t_{K-k}\cdots t_{K-l}T_{K-l+1}} \in \iota L^K_{l-2}(\Omega) \subset \Gamma_{K-l+2}\right\}, \end{aligned}$$

defined by induction on l, is non-empty.

The element g belongs to $\iota L^K_k(\Omega)$ if and only if it satisfies (1), (2), (3) and moreover:

(4) $\sigma^{-1}(1)$ belongs to the set:

$$\begin{aligned} I(\emptyset) = \left\{T_{K-k} \mid \sigma^{-1}_{T_{K-k}}(1) \in I(T_{K-k})\right\} \\ = \left\{T_{K-k} \mid g_{T_{K-k}} \in \iota L^K_{k-1}(\Omega) \subset \Gamma_{K-k+1}\right\}. \end{aligned}$$

The element g belongs to $\mathrm{Int}(L^K_k(\Omega))$ if and only if it satisfies (1), (2), (3), (4) and moreover:

(5) $g_{\tau_{K-k}\cdots\tau_K} \in \mathrm{Int}(\Omega)$.

This description and especially point (5) prove the:

Fact 4.11. — *With respect to the generating set $A_{K-k}\cup B_{K-k}$ of the group Γ_{K-k}, and the generating set B_{K+1} of the group H_{K+1}, one has:*

$$|\operatorname{Int}(L_k^K(\Omega))| = |\iota L_k^K(\Omega)|\frac{|\operatorname{Int}(\Omega)|}{|\Omega|}.$$

In particular, the set $\iota L_k^K(\Omega)$ is precisely the interior $\operatorname{Int}(L_k^K(\Omega))$ when $\operatorname{Int}(\Omega)=\Omega$. This happens when H_{K+1} (hence H_0) is finite.

For $0\leqslant k\leqslant K$, set $\frac{|\iota L_k^K(\Omega)|}{|L_k^K(\Omega)|}=1-\varepsilon_k$. The number ε_k will be denoted ε_k^K later on to emphasize the dependance on K. Lemma 3.7 generalizes as:

Lemma 4.12. — *The sequence $(\varepsilon_k)_{0\leqslant k\leqslant K}$ satisfies $\varepsilon_0 = 1-\frac{1}{d_K}$ and:*

$$1-\varepsilon_{k+1} = \frac{1-\varepsilon_k}{1-\varepsilon_k^{d_{K-k-1}}}.$$

Proof. — Given a subset $I\subset\{1,\dots,d_{K-k-1}\}$, denote:

$$J_I = \big\{g=(g_1,\dots,g_{d_{K-k-1}})\sigma \mid \forall T\in I, g_T\in \iota L_k^K(\Omega) \\ \text{and } \forall t\notin I, g_t\in L_k^K(\Omega)\smallsetminus \iota L_k^K(\Omega)\big\}.$$

By definition, $L_{k+1}^K(\Omega)$ is the disjoint union $L_{k+1}^K(\Omega)=\sqcup_{|I|\geqslant 1} J_I$.

As in the proof of Lemma 3.7, one has for $i=|I|$:

$$|J_I| = |A_{K-k-1}||L_k^K(\Omega)|^{d_{K-k-1}}(1-\varepsilon_k)^i\varepsilon_k^{d_{K-k-1}-i},$$
$$|J_I\cap \iota L_{k+1}^K(\Omega)| = \frac{i}{d_{K-k-1}}|J_I|.$$

Again by use of the mean of binomial distribution, get:

$$\begin{aligned}|\iota L_{k+1}^K(\Omega)| &= \sum_{i=1}^{d_{K-k-1}} C^i_{d_{K-k-1}}(1-\varepsilon_k)^i\varepsilon_k^{d_{K-k-1}-i}\frac{i}{d_{K-k-1}}|L_k^K(\Omega)|^{d_{K-k-1}}|A_{K-k-1}| \\ &= (1-\varepsilon_k)|L_k^K(\Omega)|^{d_{K-k-1}}|A_{K-k-1}|,\end{aligned}$$

$$\begin{aligned}|L_{k+1}^K(\Omega)| &= \sum_{i=1}^{d_{K-k-1}} C^i_{d_{K-k-1}}(1-\varepsilon_k)^i\varepsilon_k^{d_{K-k-1}-i}|L_k^K(\Omega)|^{d_{K-k-1}}|A_{K-k-1}| \\ &= (1-\varepsilon_k^{d_{K-k-1}})|L_k^K(\Omega)|^{d_{K-k-1}}|A_{K-k-1}|.\end{aligned}$$

This proves the lemma. □

Lemma 4.13. — *If $\frac{d_k}{\log k}\longrightarrow 0$, then $\varepsilon_K^K\longrightarrow 0$. If $d_k\leqslant D$ for all k, then $\varepsilon_K^K = O(K^{-\eta})$ for all $\eta<\frac{1}{D-1}$.*

First check the elementary:

Fact 4.14. — *Let* $f(D,\varepsilon)=\frac{1-\varepsilon^{D-1}}{1-\varepsilon^{D}}$, *for* $D\geqslant 2$ *and* $\varepsilon\in(0,1)$. *Then for fixed* D, *the function* $f(D,\varepsilon)$ *is decreasing with* ε, *and for fixed* ε, *the function* $f(D,\varepsilon)$ *is increasing with* D.

Proof. — Compute derivatives:

$$(1-\varepsilon^{D})^2\frac{\partial f}{\partial\varepsilon}(D,\varepsilon)=\varepsilon^{D-2}(1-\varepsilon)\big(\varepsilon^{D-1}+\cdots+\varepsilon^2+\varepsilon-(D-1)\big)<0,$$
$$(1-\varepsilon^{D})^2\frac{\partial f}{\partial D}(D,\varepsilon)=\varepsilon^{D-1}(\varepsilon-1)\log\varepsilon>0.$$

□

Proof of Lemma 4.13. — For a fixed K, and $0\leqslant k\leqslant K$, set $D_k=d_{K-k}$, and $D(K)=\max_{0\leqslant k\leqslant K}\{d_k\}=o(\log K)$. By Lemma 4.12, one has:

$$\varepsilon_{k+1}=\varepsilon_k\frac{1-\varepsilon_k^{D_{k+1}-1}}{1-\varepsilon_k^{D_{k+1}}}=\varepsilon_k f(D_{k+1},\varepsilon_k).$$

By fact 4.14, as long as $\varepsilon_k\geqslant E$, one has:

$$\varepsilon_{k+1}\leqslant\varepsilon_k f(D_{k+1},E)\leqslant\varepsilon_k f(D(K),E),$$

so $\varepsilon_K=\varepsilon_K^K\leqslant\max\{E,f(D(K),E)^K\}$ for any $E\in(0,1)$. Now consider a sequence $E_K\longrightarrow 0$ so that $|D(K)\log E_K|=o(\log K)$ (it exists). One has:

$$\begin{aligned}f(D(K),E_K)^K&=\exp K\big(\log(1-E_K^{D(K)-1})-\log(1-E_K^{D(K)})\big),\\&=\exp\big(-KE_K^{D(K)-1}+O(KE_K^{D(K)})\big)\longrightarrow 0,\end{aligned}$$

because $KE_K^{D(K)-1}\longrightarrow+\infty$. This shows $\varepsilon_K^K\longrightarrow 0$.

If moreover $d_k\leqslant D$, take $E_K=K^{-\eta}$ with $\eta<\frac{1}{D-1}$, then:

$$f(D,E_K)^K=\exp\big(-K^{1-\eta(D-1)}+O(K^{1-\eta D})\big)=o(K^{-\eta}),$$

so $\varepsilon_K^K=O(K^{-\eta})$. □

Proof of Theorem 4.6. — By Fact 4.11, one has:

$$\frac{|\operatorname{Int}(L_K^K(\Omega))|}{|L_K^K(\Omega)|}=\frac{|\iota L_K^K(\Omega)|}{|L_K^K(\Omega)|}\frac{|\operatorname{Int}(\Omega)|}{|\Omega|}=(1-\varepsilon_K^K)\frac{|\operatorname{Int}(\Omega)|}{|\Omega|}.$$

As the group H_{K+1} is amenable by Fact 4.4, the set Ω can be chosen with $\frac{|\operatorname{Int}(\Omega)|}{|\Omega|}$ arbitrarily close to 1. By Lemma 4.13, this shows that there exists a sequence of sets $\Omega_K\subset H_{K+1}$ so that the sets $L_K^K(\Omega_K)\subset\Gamma_0$ form a Folner sequence. □

5. Examples of groups with property $\mathcal{DP}$

5.1. Alternate directed groups

Given a sequence $\bar{d} = (d_i)_{i\in\mathbb{N}}$ of integers $d_i \geqslant 2$, set:

$$AT_i = AT(d_i, d_{i+1}) = (\mathcal{A}_{d_{i+1}} \times \cdots \times \mathcal{A}_{d_{i+1}}) \rtimes \mathcal{A}_{d_i-1} = \mathcal{A}_{d_{i+1}} \wr \mathcal{A}_{d_i-1},$$

where $\mathcal{A}_d$ is the alternate group of even permutations of the set $\{1, \ldots, d\}$, there are $d_i - 1$ factors in the product (indexed by $\{2, \ldots, d_i\}$), and $\mathcal{A}_{d_i-1}$ acts by permuting these factors. Consider the countable infinite direct product:

$$H_{\bar{d}}^{\mathrm{alt}} = \prod_{i=0}^{\infty} AT_i = \prod_{i=0}^{\infty} \mathcal{A}_{d_{i+1}} \wr \mathcal{A}_{d_i-1}.$$

Its elements are denoted as sequences $h = (h_i)_{i=0}^{\infty}$ with $h_i = (a_{i,2}, \ldots, a_{i,d_i})\rho_i \in AT_i$.

The group $H_{\bar{d}}^{\mathrm{alt}}$ acts faithfully on the spherically homogeneous rooted tree $T_{\bar{d}}$ in the direction of the ray 1^∞, where under the canonical isomorphism (2.1), one has:

$$(h_i)_{i=0}^{\infty} = ((h_i)_{i=1}^{\infty}, a_{0,2}, \ldots, a_{0,d_0})\rho_0,$$

where $\rho_0 \in \mathcal{A}_{d_0-1} \simeq \mathrm{Fix}_{\mathcal{A}_{d_0}}(1)$. Inductively under isomorphism $\mathrm{Aut}(T_{s^k\bar{d}}) \simeq \mathrm{Aut}(T_{s^{k+1}\bar{d}}) \wr S_{d_k}$, one has $(h_i)_{i=k}^{\infty} = ((h_i)_{i=k+1}^{\infty}, a_{k,2}, \ldots, a_{k,d_k})\rho_k$.

On the other hand, the group $\mathcal{A}_{d_0}$ acts on $T_{\bar{d}}$ by rooted automorphisms:

$$\mathcal{A}_{d_0} \ni a = (e, \ldots, e)a.$$

Definition 5.1. — *An alternate directed group G is a subgroup of $\mathrm{Aut}^{\mathrm{alt}}(T_{\bar{d}})$ with generating set $A \cup H$, with $A \subset \mathcal{A}_{d_0}$ and $H \subset H_{\bar{d}}^{\mathrm{alt}}$. Denote:*

$$G(A, H) = \langle A \cup H \rangle < \mathrm{Aut}^{\mathrm{alt}}(T_{\bar{d}}).$$

When the sequence $\bar{d}$ is constant $d_i = d$, if $A \simeq \mathcal{A}_d$ and $H \simeq \mathcal{A}_d \wr \mathcal{A}_{d-1}$ is diagonaly embedded into the direct product $H_{\bar{d}}^{\mathrm{alt}}$, then $G(A, H) = G_d$ is the alternate mother group of section 3. Directed groups (not necessarily alternate) satisfy the same definition without requirement that the permutations involved are even, that is with S_d instead of $\mathcal{A}_d$ and $H_{\bar{d}} = \prod_{i=0}^{\infty} S_{d_{i+1}} \wr S_{d_i-1}$ instead of $H_{\bar{d}}^{\mathrm{alt}}$ (see [6], [7]).

5.2. Case of bounded valency

In this section, assume that the sequence $\bar{d}$ is bounded $5 \leqslant d_i \leqslant D$. Let $B \subset H_{\bar{d}}^{\mathrm{alt}}$ be a finite subset, and denote its elements by $\beta = (\beta_i)_{i=0}^{\infty} \in H_{\bar{d}}^{\mathrm{alt}}$. Then for each i, the set $\{\beta_i, \beta \in B\}$ is a B-indexed subset of $AT_i = AT(d_i, d_{i+1})$. As the valency sequence $\bar{d}$ is bounded, there is a finite set of pairs:

$$\big\{(AT(s), \{\beta(s), \beta \in B\}), s \in J\big\},$$

such that for any i, there exists $s(i)$ in the finite set J with $(AT_i, \{\beta_i, \beta \in B\}) = (AT(s(i)), \{\beta(s(i)), \beta \in B\})$, as pairs of finite groups with B-indexed subsets.

This provides an isomorphism:

$$H_{\bar{d}}^{\mathrm{alt}} > H = \langle \beta, \beta \in B \rangle \simeq \langle (\beta(s))_{s \in J}, \beta \in B \rangle < \prod_{s \in J} AT(s).$$

The group H is said saturated if $H = \prod_{s \in J} AT(s)$. (Mind a difference with the notion of saturation in [6] and [7], where it was only required that H surjects on each factor $AT(s)$. The present condition is slightly stronger.) Finiteness of J shows the:

Fact 5.2. — *If $\bar{d}$ is bounded, any finitely generated subgroup of $H_{\bar{d}}^{\mathrm{alt}}$ is contained in a finite saturated subgroup H.*

The following proposition will permit to show amenability of all directed groups acting on a tree of bounded valency.

Proposition 5.3. — *Let $\bar{d}$ be a bounded sequence of integers $d_i \geqslant 5$. If $H < H_{\bar{d}}^{\mathrm{alt}}$ is a finite saturated subgroup, then the alternate directed group $G(\mathcal{A}_{d_0}, H) < \mathrm{Aut}(T_{\bar{d}})$ has property $\mathcal{DP}$.*

Proof. — Set $A_k = \mathcal{A}_{d_k}$, $H_k = \{(h_i)_{i=k}^{\infty} \mid (h_i)_{i=0}^{\infty} \in H\}$, define $\Gamma_k = G(A_k, H_k) < \mathrm{Aut}^{\mathrm{alt}}(T_{s^k \bar{d}})$ and check that the sequence $\{\Gamma_k\}_{k \in \mathbb{N}}$ has property $\mathcal{DP}$. In order to ease notations, we treat the case $k = 0$, the general case is similar.

The only non-trivial point in order to verify the conditions of Definition 4.1 is surjectivity of the isomorphism:

$$\varphi_0 \colon G(\mathcal{A}_{d_0}, H) \longrightarrow G(\mathcal{A}_{d_1}, H_1) \wr \mathcal{A}_{d_0}.$$

Given $h = (h_i)_{i=0}^{\infty}$ in $H_{\bar{d}}$ with $h_i = (a_{i,2}, \ldots, a_{i,d_i})\rho_i$, set:

$$h(2) = ((a_{i,2}, e, \ldots, e)e)_{i=0}^{\infty}, \text{ and } h(\emptyset) = ((e, \ldots, e)\rho_i)_{i=0}^{\infty}.$$

In each factor $AT(s) = \mathcal{A}_{d'(s)} \wr \mathcal{A}_{d(s)-1}$, the subset

$$\{(a_2, e, \ldots, e) \mid a_2 \in \mathcal{A}_{d'(s)}\} \cup \{(e, \ldots, e)\rho \mid \rho \in \mathcal{A}_{d(s)-1}\}$$

generates the group $AT(s)$. Thus by saturation

$$\langle h(2), h\in H\rangle \simeq \prod_{s\in J}\mathcal{A}_{d'(s)}\times\{e\}\times\cdots\times\{e\},\ \text{and}\ \langle h(\emptyset), h\in H\rangle \simeq \prod_{s\in J}\mathcal{A}_{d'(s)}.$$

So saturation shows that the subsets $H(2)=\{h(2), h\in H\}$ and $H(\emptyset)=\{h(\emptyset), h\in H\}$ are subgroups of H, and moreover $\langle H(2)\cup H(\emptyset)\rangle = H$.

The proofs of Fact 3.2 and Proposition 3.1 apply directly, replacing the generators $b_2=b(\alpha_2, e,\dots,e,e_A)$ and $b_\emptyset = b(e,\dots,e,\rho)$ by $h(2)$ and $h(\emptyset)$ respectively. □

Let σ be a permutation of the set $\{1,\dots,d\}$. Denote σ' another copy of σ acting on the set $\{d+1,\dots,2d\}$ by $\sigma'(t)=\sigma(t-d)+d$, and consider the embedding $a\colon S_d\hookrightarrow \mathcal{A}_{2d}$ given by $a(\sigma)=\sigma\sigma'$. It can be extended to furnish:

$$a\colon \operatorname{Aut}(T_{\bar d})\to \operatorname{Aut}^{\text{alt}}(T_{\bar{2d}}),$$

an embedding of the group of automorphisms of the tree $T_{\bar d}$ into the group of alternate automorphisms of the tree $T_{\bar{2d}}$.

Indeed, let $\gamma\in\operatorname{Aut}(T_{\bar d})$ be described by a family of permutations $\{\sigma_v\}_{v\in T_{\bar d}}$, where $\sigma_v\in S_{d_k}$ for every $v=t_1\cdots t_k$ in $T_{\bar d}$. The automorphism $a(\gamma)$ is described by a family of permutations $\{a(\gamma)_v\}_{v\in T_{\bar{2d}}}$ given by $a(\gamma)_v=a(\gamma_v)\in\mathcal{A}_{2d_k}$ for $v=t_1\cdots t_k$ in $T_{\bar d}\subset T_{\bar{2d}}$ and $a(\gamma)_v=e$ for $v\in T_{\bar{2d}}\smallsetminus T_{\bar d}$.

Fact 5.4. — *Directed elements have directed image under a, i.e. $a(H_{\bar d})\subset H^{\text{alt}}_{\bar{2d}}$. In particular, the mother group of degree 0 acting on a d-regular tree embeds in the alternate mother group G_{2d} acting on a $2d$-regular tree.*

Proof. — As a shortcut denote 1^k for the sequence $11\cdots1$ with k ones. By definition, an automorphism γ is directed if and only if $\sigma_{1^k}\in\operatorname{Fix}_{S_{d_k}}(1)\simeq S_{d_k-1}$ and $\sigma_v=e$ if v is not of the form $1^{k-1}t$ for some t in $\{1,\dots,d_k\}$. This is still the case for $a(\gamma)$. □

The following result from [6] can now be reproved.

Corollary 5.5. — *Directed groups acting on a tree of bounded valency are amenable.*

Proof. — Let Γ be a directed group, with generating set $S\cup H$ where $S\subset S_{d_0}$ and $H\subset H_{\bar d}$. By fact 5.4, the group $a(\Gamma)<\operatorname{Aut}^{\text{alt}}(T_{\bar{2d}})$ is alternate and directed. By fact 5.2, it can be included in a directed, alternate and saturated subgroup of $\operatorname{Aut}^{\text{alt}}(T_{\bar{2d}})$, which has property $\mathcal{DP}$ by Proposition 5.3, hence $a(\Gamma)$ is amenable by Theorem 4.6, since $\bar{2d}$ is bounded and H_0 finite. The group Γ is also amenable as a subgroup. □

Corollary 5.6 (Main theorem in [3]). — *Automata groups with bounded activity are amenable.*

Proof. — By Theorem 3.3 in [3], an automata group Γ with bounded activity is a subgroup of the alternate mother group of degree 0 acting on a d-regular tree for d large enough. By Fact 5.4, Γ is a subgroup of G_{2d}, hence is amenable by Theorem 3.3. □

5.3. Examples with unbounded valency

This section aims at constructing examples of groups with property $\mathcal{DP}$ for which the sequence $\bar{d}$ of fact 4.2 is unbounded.

Let H be a finitely generated, residually finite, perfect group with a sequence of normal subgroups $(N_i)_{i\geqslant 0}$ of finite index so that each quotient $A_i = H/N_i$ is perfect, acting faithfully and transitively on a finite set $\{1,\dots,d_i\}$ of size $d_i \geqslant 2$. For h in H, denote $a_i(h) = hN_i \in A_i$. Assume moreover that there exists $\tau_i \in A_i$ such that $\tau_i(1) = 1$ and $\tau_i^{-1}(2) \notin \{1,2\}$.

To the group $H = H_0$ together with subgroup sequence $(N_k)_{k\geqslant 0}$ is associated an action on the rooted tree $T_{\bar{d}}$ of valency sequence $\bar{d} = (d_k)_{k\geqslant 0}$, denoted $b_0\colon H_0 \to \mathrm{Aut}(T_{\bar{d}})$, given by the portrait $(b_0(h))_{1^{k-1}2} = a_k(h)$ and $(b_0(h))_v = e$ if v is not of the form $1^{k-1}2$ for $k \geqslant 1$ (notation 1^k is a shortcut for $11\cdots1$ with k ones).

More generally, to $H_i = H$ together with the sequence $(N_k)_{k\geqslant i}$ is associated an action on $T_{s^i\bar{d}}$ denoted $b_i\colon H_i \to \mathrm{Aut}(T_{s^i\bar{d}})$, given by the portrait $(b_i(h))_{1^{k-1}2} = a_{k+i}(h)$ and $(b_i(h))_v = e$ if v is not of the form $1^{k-1}2$.

The group $A_i = H_i/N_i = H/N_i$ also acts on $T_{s^i\bar{d}}$ as a rooted automorphism acting on $\{1,\cdots,d_i\}$, *i.e.* $a_i(h) = (e,\dots,e)a_i(h)$. Set $\Gamma_i = \langle A_i \cup H_i\rangle < \mathrm{Aut}(T_{s^i\bar{d}})$, with the b_i action.

Fact 5.7. — *The sequence of groups $\{\Gamma_i\}_{i\in\mathbb{N}}$ has property $\mathcal{DP}$.*

Proof. — In the wreath product isomorphism φ_i of (2.1) for $T_{s^i\bar{d}}$, one has:

$$\varphi_i(b_i(h)) = \big(b_{i+1}(h), a_{i+1}(h), e, \dots, e\big). \tag{5.1}$$

Thus it induces an embedding $\varphi_i\colon \Gamma_i \hookrightarrow \Gamma_{i+1} \wr A_i$. The only non-trivial point in the conditions of Definition 4.1 is to check that this embedding is onto.

As $\varphi_i(b_i(h')^{\tau_i}) = (b_{i+1}(h'), e, \dots, a_{i+1}(h'), \dots, e)$, one has $\varphi_i([b_i(h')^{\tau_i}, b_i(h)]) = (b_{i+1}([h',h]), e, \dots, e)$. As H is perfect, the image contains $H_{i+1} \times \{e\} \times \cdots \times \{e\}$, and also $\{e\} \times A_{i+1} \times \{e\} \times \cdots \times \{e\}$ by (5.1). Moreover, the image contains A_i rooted which has a transitive action on $\{1, \dots, d_i\}$. Thus $\varphi_i(\Gamma_i)$ finally contains $\Gamma_{i+1} \wr A_i$. □

As an example of such a finitely generated, residually finite, perfect group H, one may take the alternate mother group G_d of section 3 for $d \geqslant 6$ (for which both finite generating subgroups A and B are perfect). This group satisfies $G_d \simeq G_d \wr \mathcal{A}_d$. Its finite index normal subgroups are:

$$St_j = \ker(G_d \to \mathcal{A}_d \wr \cdots \wr \mathcal{A}_d),$$

where the j factors in the iterated wreath product are obtained by iteration of the above isomorphism. The group St_j is called stabilizer of level j of the group G_d. The quotient G_d/St_j is acting transitively on level j, which is the set $\{1, \dots, d\}^j$. By [15], these stabilizers St_j are the only finite index normal subgroups of G_d.

For an arbitrary function $j\colon \mathbb{N} \to \mathbb{N}$, take $N_k = St_{j(k)}$ as a sequence of normal subgroups. The group Γ_0 defined by $H = G_d$ together with the function $j(k)$ has property $\mathcal{DP}$ by Fact 5.7. It is amenable when $d^{j(k)}$ is sublogarithmic by Theorem 4.6. Note that in the construction above, one could use any group of Proposition 5.3 with $d_i \geqslant 6$ instead of G_d.

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BIBLIOGRAPHY

[1] S. Aleshin, "Finite automata and Burnside's problem for periodic groups", *Math. Notes* **11** (1972), p. 199-203.

[2] G. Amir, O. Angel & B. Virág, "Amenability of linear-activity automaton groups", *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 3, p. 705-730.

[3] L. Bartholdi, V. A. Kaimanovich & V. V. Nekrashevych, "On amenability of automata groups", *Duke Math. J.* **154** (2010), no. 3, p. 575-598.

[4] L. Bartholdi & B. Virág, "Amenability via random walks", *Duke Math. J.* **130** (2005), no. 1, p. 39-56.

[5] J. Brieussel, "Growth behaviors in the range e^{r^α}", Preprint arXiv: 1107.1632 to appear in Afrika Matematika.

[6] ———, "Amenability and non-uniform growth of some directed automorphism groups of a rooted tree", *Math. Z.* **263** (2009), no. 2, p. 265-293.

[7] ———, "Behaviors of entropy on finitely generated groups", *Ann. Probab.* **41** (2013), no. 6, p. 4116-4161.

[8] A. Erschler, "Isoperimetry for wreath products of Markov chains and multiplicity of selfintersections of random walks", *Probab. Theory Related Fields* **136** (2006), no. 4, p. 560-586.

[9] E. Følner, "On groups with full Banach mean value", *Math. Scand.* **3** (1955), p. 243-254.

[10] R. I. Grigorchuk, "Degrees of growth of finitely generated groups and the theory of invariant means", *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 5, p. 939-985.

[11] A. Grigor′yan, "Heat kernel upper bounds on a complete non-compact manifold", *Rev. Mat. Iberoamericana* **10** (1994), no. 2, p. 395-452.

[12] M. Gromov, "Groups of polynomial growth and expanding maps", *Inst. Hautes Études Sci. Publ. Math.* (1981), no. 53, p. 53-73.

[13] V. A. Kaĭmanovich & A. M. Vershik, "Random walks on discrete groups: boundary and entropy", *Ann. Probab.* **11** (1983), no. 3, p. 457-490.

[14] H. Kesten, "Full Banach mean values on countable groups", *Math. Scand.* **7** (1959), p. 146-156.

[15] P. M. Neumann, "Some questions of Edjvet and Pride about infinite groups", *Illinois J. Math.* **30** (1986), no. 2, p. 301-316.

[16] P. Pansu, "Croissance des boules et des géodésiques fermées dans les nilvariétés", *Ergodic Theory Dynam. Systems* **3** (1983), no. 3, p. 415-445.

[17] C. Pittet & L. Saloff-Coste, "Random walks on finite rank solvable groups", *J. Eur. Math. Soc. (JEMS)* **5** (2003), no. 4, p. 313-342.

[18] J. S. Wilson, "Further groups that do not have uniformly exponential growth", *J. Algebra* **279** (2004), no. 1, p. 292-301.

[19] ———, "On exponential growth and uniformly exponential growth for groups", *Invent. Math.* **155** (2004), no. 2, p. 287-303.

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