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The Dyson Brownian Minor Process

THE DYSON BROWNIAN MINOR PROCESS

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ABSTRACT. — Consider an $n \times n$ Hermitean matrix valued stochastic process $\{H_t\}_{t \geqslant 0}$ where the elements evolve according to Ornstein-Uhlenbeck processes. It is well known that the eigenvalues perform a so called Dyson Brownian motion, that is they behave as Ornstein-Uhlenbeck processes conditioned never to intersect.

In this paper we study not only the eigenvalues of the full matrix, but also the eigenvalues of all the principal minors. That is, the eigenvalues of the $k \times k$ minors in the upper left corner of H_t. Projecting this process to a space-like path leads to a determinantal process for which we compute the kernel. This kernel contains the well known GUE minor kernel, and the Dyson Brownian motion kernel as special cases.

In the bulk scaling limit of this kernel it is possible to recover a time-dependent generalisation of Boutillier's bead kernel.

We also compute the kernel for a process of intertwined Brownian motions introduced by Warren. That too is a determinantal process along space-like paths.

RÉSUMÉ. — Nous considérons une processus stochastique fourni par une matrice $\{H_t\}_{t \geqslant 0}$ de taille n, dont les éléments évoluent selon un processus d'Ornstein-Uhlenbeck. Les valeurs propres de H_t évoluent selon un mouvement Brownien de Dyson, c'est-à-dire qu'elles décrivent n processus d'Ornstein-Uhlenbeck répulsifs.

Dans cet article, nous considérons non seulement les valeurs propres de la matrice elle-même, mais aussi les valeurs propres combinées avec celles des mineurs principaux ; c'est-à-dire les valeurs propres des sous-matrices dans le coin supérieur gauche de la matrice H_t. Ce processus, projeté sur des chemins "spatiaux" appropriés, est un processus déterminantal dont nous fournissons le noyau ; en outre, le noyau GUE-mineur et le noyau du processus de Dyson apparaissent tous deux comme des cas particuliers.

La limite dans le "bulk" de ce noyau fournit une généralisation, dépendante du temps, du noyau "bead" de Boutillier.

Nous calculons également le noyau pour un processus de mouvements browniens entrelacés introduit par Warren ; celui-ci est également un processus déterminantal le long de chemins spatiaux.

1. Introduction

In a classic paper Dyson [15] introduced a dynamics on random Hermitean matrices where each free matrix element evolves independently of all others. They each form an Ornstein-Uhlenbeck process, that is a Brownian motion with a drift toward zero. He successfully analysed the associated dynamics of the eigenvalues. Even the fact that the eigenvalues form a Markov process is highly non-trivial.

This has become one of the most well studied models of random matrix theory. It is beyond the scope of this paper to completely survey the literature but some notable results are these. It has been found to be a limit of discrete random walks conditioned never to intersect, for which correlation kernels have been found; see [33, 20, 21, 23, 22, 16, 28, 29]. Other work includes [4, 37, 38, 1]. In particular, in [38, 1], partial differential equations were derived for the Dyson process and related processes. Actually there are two processes that have been called Dyson Brownian motion. The first is this: Let a Hermitean matrix B_t evolve according to the transition density, for $s < t$,

$$(1.1) \qquad \Pr(B_t \in dB | B_s = \bar{B}) = Ce^{-Tr(B - q_{t-s}\bar{B})^2/(1-q_{t-s}^2)}$$

where $q_{t-s} = e^{-(t-s)}$ and C is the normalisation constant that makes this a probability density. Then this is a stationary process and its stationary measure is called the *Gaussian Unitary Ensemble* (GUE), see [32, chapter 9]. An $n \times n$ GUE matrix B is given by

$$\Pr(B \in dB) = Ce^{-\operatorname{Tr} B^2} dB.$$

What this boils down to is that the elements on the diagonal are independent Gaussians with mean 0 and variance $\frac{1}{2}$. For the off-diagonal elements the real and imaginary parts are independent Gaussians with mean 0 and variance $\frac{1}{4}$. For an $n \times n$ Hermitean matrix B let $\operatorname{eig} B = (\lambda_1 < \cdots < \lambda_n)$

denote the vector of eigenvalues of B. The distribution of the eigenvalues is exactly

$$\Pr(\text{eig } B \in d\lambda) = C \prod_{1 \leqslant i < j \leqslant n} (\lambda_i - \lambda_j)^2 \prod_{i=1}^n e^{-\lambda_i^2} \, d\lambda_i$$

where C is a normalisation constant, see [32, chapter 3]. For (1.1) the transition density for the eigenvalues is then, by the Harish-Chandra formula,

$$\Pr(\text{eig } B_t \in d\lambda \,|\, \text{eig } B_s = \bar{\lambda})$$
$$= C \frac{\Delta(\lambda)}{\Delta(\bar{\lambda})} \det[e^{-(\lambda_i - e^{-(t-s)}\bar{\lambda}_j)^2/(1-e^{-2(t-s)})}]_{i,j=1}^n \prod_{i=1}^n d\lambda_i.$$

Here, Δ denotes the Vandermonde determinant. See [23] for a readable overview. This expression is a Doob h-transform of a Karlin-McGregor determinant, see [14, 27]. That means that this can be interpreted probabilistically as n Ornstein-Uhlenbeck processes evolving in time conditioned never to intersect.

A second process that is often called Dyson Brownian Motion, but is not stationary, is the following. Consider n standard Brownian motions $(x_1(t), \ldots, x_n(t))_{t \in \mathbb{R}}$ conditioned never to intersect. The transition density for that process would be

$$(1.2) \quad \Pr(x(t) \in dx \,|\, x(s) = \bar{x}) = C \frac{\Delta(x)}{\Delta(\bar{x})} \det[e^{-(x_i - \bar{x}_j)^2/(t-s)}]_{i,j=1}^n \prod_{i=1}^n dx_i.$$

It can be realised by a Hermitean matrix where the elements evolve as identical independent standard Brownian motions.

For the purpose of this article consider the following model which we shall call the *Dyson Brownian minor process* or (DBM process). Let $(B_t)_{t \in \mathbb{R}^+}$ be an $N \times N$ Hermitean matrix-valued stochastic process started at $t = 0$ with B_0 given by the GUE distribution. Let the process evolve with transition density given by (1.1). For $n = 1, \ldots, N$ let $B_t^{(n)}$ be the $n \times n$ submatrix in the upper left corner (principal minor) of B_t. We are interested in all the $\binom{N+1}{2}$ eigenvalues of $B_t^{(n)}$ for $n = 1, \ldots, N$.

If λ is an eigenvalue of $B_t^{(n)}$ then we shall say that there is a particle at (n, λ, t). In this way we can think of the DBM process as a point process, that is a measure on configurations of points or particles on the space $\{1, \ldots, N\} \times \mathbb{R} \times \mathbb{R}^+$. It turns out that this is, in the terminology of [6], a determinantal point process along space-like paths. More precisely, that means the following.

For notation we shall write that

$$(n, t) < (n', t') := \begin{cases} \texttt{true} & \text{if } n > n' \\ \texttt{true} & \text{if } n = n' \text{ and } t < t' \\ \texttt{false} & \text{otherwise.} \end{cases}$$

and

$$(n, t) \geqslant (n', t') := \neg((n, t) < (n', t')).$$

THEOREM 1.1. — *Take a sequence* $\{(n_i, x_i, t_i)\}_{i=1}^k$ *of levels, positions and times. Let them follow a space-like path, which means that*

(1.3) $$0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k,$$

(1.4) $$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k.$$

Then the density of the event that there is a particle at time t_i *on level* n_i *at position* x_i *in the Dyson Brownian minor process is*

$$\rho((n_1, x_1, t_1), \ldots, (n_k, x_k, t_k)) = \det[K^{\mathcal{DBM}}((n_i, x_i, t_i), (n_j, x_j, t_j))]_{i,j=1}^k$$

where

$$K^{\mathcal{DBM}}((n, x, t), (n', x', t')) =$$

$$\begin{cases} \displaystyle\sum_{l=-\infty}^{-1} \sqrt{\frac{(n'+l)!}{(n+l)!}} e^{-l(t'-t)} h_{n+l}^*(x) h_{n'+l}^*(x') e^{-(x')^2}, & \text{for } (n, t) \geqslant (n', t'), \\ \displaystyle -\sum_{l=0}^{\infty} \sqrt{\frac{(n'+l)!}{(n+l)!}} e^{-l(t'-t)} h_{n+l}^*(x) h_{n'+l}^*(x') e^{-(x')^2}, & \text{for } (n, t) < (n', t'). \end{cases}$$

Furthermore

(1.5) $$K^{\mathcal{DBM}}((n, x, t), (n', x', t')) = -\phi^{\mathcal{DBM}}((n, x, t), (n', x', t'))$$

$$+ 2^{(n'-n)/2} \frac{2}{(2\pi i)^2} \int_\gamma du \int_\Gamma dv \frac{v^{n'}}{u^n} \frac{e^{-u^2+2ux+v^2-2vx'}}{e^{-(t'-t)}v - u}$$

where

(1.6) $$\phi^{\mathcal{DBM}}((n, x, t), (n', x', t')) =$$

$$\begin{cases} 2^{(n-n')/2} e^{n'(t'-t)} \displaystyle\int_\mathbb{R} H^{n-n'}(x - y) p_{t'-t}^*(y, x') \, dy, & \text{if } (n, t) < (n', t'), \\ 0, & \text{otherwise.} \end{cases}$$

The contours of integration are such that γ *encloses the pole at the origin and* Γ *goes from* $-i\infty$ *to* $i\infty$ *in such a way that* $|u| < |v|$ *always, see Figure 1.1.*
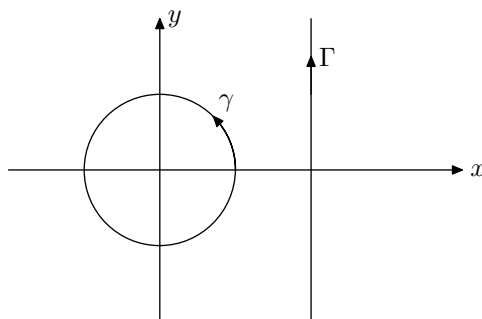
Figure 1.1. *Contours of integration.*

Here the h_n^*, for $n = 0, 1, \ldots$, are the normalised Hermite polynomials of order n, see Section 6.1. The h_n^* for $n = -1, -2, \ldots$, are defined to be zero. The function p^* is the transition density of an Ornstein-Uhlenbeck process, see (6.2). Also H^n is the nth anti-derivative of the Dirac delta function, see (6.1). Theorem 1.1 will be proved in Section 7.5.

The term space-like path for a path in space-time satisfying (1.3) and (1.4) was coined in [6] and, while the reason for using that name is not made clear, the terminology has become standard.

It is quite clear that this process, along a space-like path, is a Markov process. It is known from Dyson [15] that this process is Markovian on a fixed level, that is for constant n. Baryshnikov [3] observed that it is a Markov process for fixed time going down one level (say from n to $n-1$). A space-like path is a combination of steps in time and steps going down one level, *i.e.* it is a combination of consecutive Markov steps.

A beautiful construction due to Warren [39] goes as follows. Start a 1-dimensional Brownian motion, say $(x(t))_{t \geqslant 0}$ at the origin at time $t = 0$. Then start two new processes, say $(y_1(t))_{t \geqslant 0}$ and $(y_2(t))_{t \geqslant 0}$, such that $y_2(t) < x(t) < y_1(t)$. They evolve as Brownian motions except that they are pushed up and down, respectively, by $x(t)$. For details see [39]. It is then a theorem that y_1 and y_2 together form a Dyson Brownian motion in the sense that their transition density is of the same form as (1.2). This procedure can be continued: one can start three processes above, between and below y_1 and y_2. These three will then be a Dyson Brownian motion of three particles and so on. This process occurs as a scaling limit in the study of a certain random tiling model [35]. To reduce the amount of numerical factors floating around we shall in this paper consider the Warren process

to be driven by Brownian motions with variance $t/2$ instead of standard Brownian motion. This is just an unimportant rescaling.

For this model we can show a result analogous to that of Theorem 1.1.

THEOREM 1.2. — *Take a sequence $\{(n_i, t_i, x_i)\}_{i=1}^k$ of times as in the previous Proposition. Then the density of the event that there is a particle at time t_i on level n_i at position x_i in Warren's process is*

$$\rho((n_1, x_1, t_1), \dots, (n_k, x_k, t_k)) = \det[K^{\mathcal{W}}((n_i, x_i, t_i), (n_j, x_j, t_j))]_{i,j=1}^k$$

where

$$K^{\mathcal{W}}((n, x, t), (n', x', t')) =$$

$$\begin{cases} \dfrac{1}{\sqrt{t}} \displaystyle\sum_{l=-\infty}^{-1} \sqrt{\dfrac{(n'+l)!}{(n+l)!}} \left(\dfrac{t}{t'}\right)^l h^*_{n+l}(x/\sqrt{t}) h^*_{n'+l}(x'/\sqrt{t'}) e^{-(x')^2/t'}, \\[4pt] \hspace{6cm} \text{for } (n, t) \geqslant (n', t'), \\[10pt] -\dfrac{1}{\sqrt{t}} \displaystyle\sum_{l=0}^{\infty} \sqrt{\dfrac{(n'+l)!}{(n+l)!}} \left(\dfrac{t}{t'}\right)^l h^*_{n+l}(x/\sqrt{t}) h^*_{n'+l}(x'/\sqrt{t'}) e^{-(x')^2/t'}, \\[4pt] \hspace{6cm} \text{for } (n, t) < (n', t'). \end{cases}$$

Furthermore

$$2^{\frac{1}{2}(n-n')} K^{\mathcal{W}}((n, x, t), (n', x', t')) = -\phi^{\mathcal{W}}((n, x, t), (n', x', t'))$$

$$+ \frac{2}{(2\pi i)^2} \frac{t^{n/2}(t')^{n'/2}}{\sqrt{t}} \int_\gamma du \int_\Gamma dv \frac{v^{n'}}{u^n} \frac{e^{-u^2 + 2ux/\sqrt{t} + v^2 - 2vx'/\sqrt{t'}}}{tv - u}$$

where

$$\phi^{\mathcal{W}}((n, x, t), (n', x', t')) =$$

$$\begin{cases} \sqrt{\dfrac{(t')^{n'}}{t^n}} \displaystyle\int_{\mathbb{R}} H^{n-n'}(x-y) p_{t'-t}(y, x') \, dy, & \text{if } (n, t) < (n', t'), \\[10pt] 0, & \text{otherwise.} \end{cases}$$

The contours of integration are such that γ encloses the pole at the origin and Γ goes from $-i\infty$ to $i\infty$ in such a way that $|u| < |v|$ always, see Figure 1.1.

Again h^*_n is the normalised Hermite polynomial of order n, see Section 6.2 and H^n is the nth anti-derivative of the Dirac delta function, see (6.1). The function p is the transition density of a Brownian motion, see (6.3). Theorem 1.2 will be proved in Section 7.5.

Note that although the kernels $K^{\mathcal{W}}$ and $K^{\mathcal{DBM}}$ are just a change of variables from each other, the underlying processes are different in an essential way. As noted in [2], the difference of the eigenvalues of successive levels are pushed apart by a constant drift when they are close. In the construction due to Warren in [39] the difference of the particles on successive levels behave like the absolute value of a Brownian motion when they are close. It is to be remarked that if the minor process was constructed with Brownian motions replacing the Ornstein-Uhlenbeck prcesses, the marginals along space-like paths would agree with the Warren process.

No article on random matrices is complete without a scaling limit, so let us do one of those. In [11], Boutillier introduced a one-parameter family of models which are point processes on $\mathbb{N} \times \mathbb{R}$. On each individual copy of $\mathbb{R}$, it specialises to a determinantal process with the sine kernel which is so prevalent in all branches of random matrix theory, see [32]. Furthermore, on successive lines the particles interlace. By that we mean that if there are particles at (n, x_1) and (n, x_2), then there is a unique particle $(n+1, y)$ such that $x_1 < y < x_2$ almost surely. As a scaling limit of $K^{\mathcal{DBM}}$ above we recover a kernel which specialises to the Boutillier Bead kernel at a fixed time. One way to interpret this is to imagine all the particles in Boutillier's model moving in time in such a way that at each fixed time the picture looks like the original Bead kernel model.

THEOREM 1.3. — *Let a be a real number on the interval $(-1, 1)$. In the bulk scaling limit around $a\sqrt{2N}$ the Dyson Brownian minor kernel converges to a time dependent Bead kernel with parameter a. More precisely,*

$$K_a^{\text{Bead}}((n, x, t), (n', x', t')) = \lim_{N \to \infty} e^{-N(t'-t)}(4N)^{\frac{1}{2}(n-n')}(2N)^{-\frac{1}{2}} \times$$

$$\times K^{\mathcal{DBM}}((N + n, \sqrt{2N}a + \frac{x}{\sqrt{2N}}, \frac{t}{2N}), (N + n', \sqrt{2N}a + \frac{x'}{\sqrt{2N}}, \frac{t'}{2N}))$$

where

$$(1.7) \quad K_a^{\text{Bead}}((n, x, t), (n', x', t')) :=$$

$$-\phi_a^{\text{Bead}}((n, x, t), (n', x', t')) + \frac{1}{2\pi i} \int_{u_-}^{u_+} u^{n'-n} e^{\frac{1}{2}(t'-t)(u^2 - 2au) + u(x-y)} \, du$$

with

(1.8) $\phi_a^{\text{Bead}}((n,x,t),(n',x',t')) :=$

$$\begin{cases} 2^{\frac{1}{2}(n-n')} \int_{\mathbb{R}} H^{n-n'}(x-y) p_{\frac{1}{2}(t'-t)}(y, x'-a(t'-t)) \, dy & \text{if } (n,t) < (n',t') \\ 0 & \text{otherwise.} \end{cases}$$

The limit holds uniformly on compact sets and the contour of integration in (1.7) is the straight line between u_- and u_+ where

(1.9) $u_\pm = a \pm i\sqrt{1-a^2}$

are two points on the unit circle.

(The topology used to define compact sets on $(\mathbb{N} \times \mathbb{R}^2)^2$ is the product topology of discrete topology on $\mathbb{N}$ and Euclidean topology on $\mathbb{R}$.)

Remark that (1.8) contains the transition density p of Brownian motion, defined in (6.3), rather than p^*, defined in (6.2). Theorem 1.3 is proved in Section 8. Remark too that, since the $K^{\mathcal{DBM}}$ kernel could only be used along space-like paths, the same is true for our time dependent Bead kernel. Finally notice that the kernel K_a^{Bead} coincides with Boutillier's kernel in [11] with parameter a in the special case $t = t'$.

Of course convergence of the kernel in this case does not imply convergence of the processes, since the kernel only says something about the behaviour on space-like paths. Nor does this theorem give any hint as to how one might construct such a dynamical version of a Bead process. However, specialising the theorem above to $t = t'$ leads to the following Corollary. Though by no means unexpected, this result has to our knowledge not previously appeared in the literature.

COROLLARY 1.4. — *The GUE Minor process, defined in [25], converges in the same bulk scaling limit to Boutillier's bead process, defined in [11].*

Proof. — The kernel $K^{\mathcal{DBM}}$ specialised to $t = t'$ is exactly the GUE Minor kernel and K_a^{Bead} with $t = t'$ is Boutillier's kernel. Uniform convergence on compacts for the kernels is necessary for process convergence.   □

The plan of the paper is to first, in Sections 2, 3 and 4, outline the necessary basic theory about point processes. This is mostly a verbose summary of [10, 7]. Section 6 is devoted to computing certain convolution equalities and setting up clever notation so that Theorems 1.1 and 1.2 can be proved at the same time, performing the computation only once. In Section 7 the actual computation is performed and the article is rounded of by the asymptotic analysis in Section 8.

The fact that the GUE minor kernel can be extended to a dynamic version in this way begs the question whether something similar can be done with the antisymmetric GUE minor kernel from [12, 13, 19]. In recent works, see [8, 30], random walks conditioned to stay in Weyl chambers of the form

$$0 < x_1 < x_2 < \cdots < x_n$$

and

$$|x_1| < x_2 < \cdots < x_n$$

and their diffusion limits have been analysed. It is reasonable to believe that such processes could be realised by Ornstein-Uhlenbeck dynamics on antisymmetric purely imaginary matrices of odd respectively even size. If so then it appears this model could be analysed with the same tools used in this paper and would lead to a result similar to Theorem 1.1 but with an antisymmetric Dyson Brownian minor kernel

$$K^{\mathcal{A-DBM}}((n,x,t),(n',x',t')) =$$

$$\begin{cases} \displaystyle\sum_{l=-\infty}^{-1} \sqrt{\frac{(n'+2l)!}{(n+2l)!}} e^{-2l(t'-t)} h_{n+2l}^*(x) h_{n'+2l}^*(x') e^{-(x')^2}, & \text{for } (n,t) \geqslant (n',t'), \\ \displaystyle-\sum_{l=0}^{\infty} \sqrt{\frac{(n'+2l)!}{(n+2l)!}} e^{-2l(t'-t)} h_{n+2l}^*(x) h_{n'+2l}^*(x') e^{-(x')^2}, & \text{for } (n,t) < (n',t'). \end{cases}$$

That is beyond the scope of this paper, but this kernel is in [9] recovered as a scaling limit in a certain model related to the totally asymmetric simple exclusion process (TASEP).

While this article was being prepared it came to the attention of the authors that Patrik Ferrari and René Frings [17] were working on related problems. They prove an analog of Theorem 1.1 for matrices whose elements evolve as Brownian motions and also for the Laguerre ensemble.

2. Point processes

Let Λ be a complete, separable metric space with some reference measure λ; for instance, $\mathbb{R}$ with the Lebesgue measure or $\mathbb{Z}$ with counting measure. Let $M(\Lambda)$ be the set of integer valued and locally finite measures on Λ. A *point process* X on Λ is a measure on $M(\Lambda)$. It is beyond the scope of this article to give a complete overview of the theory of point processes, but some results which we use are detailed here.

A point process can be represented as

$$X = \sum_{i \in I} \delta_{x_i}$$

where $(x_i)_{i \in I}$ are random variables which we shall refer to as the points or the particles of X. Think of this as a random configuration of points or particles on the space Λ. In this paper we shall only consider point processes which are simple, *i.e.* all x_i are distinct.

To work with point processes it is convenient to define the so called *correlation functions*. For $n = 1, 2, \ldots$, these are functions $\rho_n : \Lambda^n \to \mathbb{R}$. When $\Lambda = \mathbb{Z}$ and λ is the counting measure on $\mathbb{Z}$,

$$\rho_n(x_1, \ldots, x_n) = \mathbb{P}[\text{There is a particle at each position } x_i, \text{ for } i{=}1, \ldots, n].$$

When $\Lambda = \mathbb{R}$ and λ is Lebesgue measure then

$$\rho_n(x_1, \ldots, x_n) =$$
$$\lim_{\epsilon \to 0} \frac{\mathbb{P}[\text{There is a particle in each of } [x_i, x_i + \epsilon), \text{ for } i = 1, \ldots, n]}{\epsilon^n}.$$

More generally, see [24], one can define correlation functions by saying that for simple, measurable functions ϕ of bounded support, the point process satisfies

$$(2.1) \quad \mathbb{E}\left[\prod_{i \in I}(1 + \phi(x_i))\right] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^{n} \phi(y_j) \rho_n(y_1, \ldots, y_n) \, d\lambda^n(y).$$

A *determinantal point process* is a point process whose correlation functions have the special form

$$\rho_n(y_1, \ldots, y_n) = \det\left[K(y_i, y_j)\right]_{i,j=1}^{n},$$

for some function $K : \Lambda^2 \to \mathbb{C}$. This is a very special and simple situation since all information about the point process is encoded in this function K of two variables which is called the correlation kernel. Nonetheless processes of this kind are commonplace in mathematics today arising from such diverse sources as tilings with rhombuses or dominoes of regions in the plane, random walks, eigenvalues of unitarily invariant random matrices and, as shown in the next section, so called L-ensembles. Indeed, the main theorems of this paper, Theorems 1.1 and 1.2, state that certain processes are indeed determinantal point processes. Also note that with correlations functions of this form, the right hand side of of (2.1) turns out to be the definition of the Fredholm determinant of the integral operator with kernel $K(x, y)\phi(y)$, again see [24].

Asymptotic analysis of such point processes can be performed by working with the kernels only.

PROPOSITION 2.1 (Proposition 2.1 in [25]). — *Let X^1, $X^2,\ldots,X^N,\ldots$ be a sequence of determinantal point processes, and let X^N have correlation kernel K^N satisfying*

(1) *$K^N \to K$, $N \to \infty$ pointwise, for some function K,*
(2) *the K^N are uniformly bounded on compact sets in Λ^2 and*
(3) *for C compact, there exists some number $n = n(C)$ such that*

$$\det[K^N(x_i, x_j)]_{1 \leqslant i,j \leqslant m} = 0$$

if $m \geqslant n$.

Then there exists some determinantal point process X with correlation kernel K such that $X^N \to X$ weakly, $N \to \infty$.

3. Introduction to L-ensembles

This section summarises the exposition in [10]. The reader who wishes to pursue the subject of determinantal point processes will find [5, 31] illuminating.

3.1. Measure theory

Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a *discrete probability space*. That is, Ω is a finite set, $\mathcal{F} = 2^{\Omega}$ is the σ-algebra of subsets of Ω. Furthermore, $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is a measure satisfying

(1) $\mathbb{P}[\Omega] = 1$,
(2) $\mathbb{P}[\varnothing] = 0$,
(3) $E_1, E_2 \in \mathcal{F}$ and $E_1 \cap E_2 = \varnothing$ implies $\mathbb{P}[E_1 \cup E_2] = \mathbb{P}[E_1] + \mathbb{P}[E_2]$.

The last property is called additivity.

An element $E \in \mathcal{F}$ is called an *event* and $\mathbb{P}[E]$ is called the probability of E. To sample the distribution $\mathbb{P}$ means picking an element $\omega \in \Omega$. Since we are working with a finite space, $\mathbb{P}[E]$ can be decomposed as a sum over singleton sets,

$$\mathbb{P}[E] = \sum_{\omega \in E} \mathbb{P}[\{\omega\}].$$

3.2. Point processes

We now specialise and consider probability spaces of the following form. Take a finite set $\mathcal{X}$. A *point process* on $\mathcal{X}$ is a probability space $(\Omega = 2^{\mathcal{X}}, \mathbb{P}, \mathcal{F} = 2^{\Omega} = 2^{2^{\mathcal{X}}})$. To sample the point process means to pick an element of $2^{\mathcal{X}}$, *i.e.* a subset of $\mathcal{X}$. For compatibility with [10], we will use uppercase letters at the end of the alphabet, X, Y, ..., when speaking about elements of Ω.

Given an $X \in \Omega$, let $E_{\mathcal{X}}(X)$ be the event

$$\mathcal{F} \ni E_{\mathcal{X}}(X) = \bigcup_{X \subseteq Y \subseteq \mathcal{X}} \{Y\}.$$

The probabilistic interpretation is that $E_{\mathcal{X}}(X)$ is the event that all $x \in X$ are in the chosen set. Note that $E_{\mathcal{X}} : \Omega \to \mathcal{F}$.

3.3. L-ensembles

Let us specialise even more. Take a matrix L of size $|\mathcal{X}| \times |\mathcal{X}|$. We shall index the rows and columns of this matrix by $\mathcal{X}$. The *L-ensemble* on $\mathcal{X}$ is a point process $(\Omega = 2^{\mathcal{X}}, \mathbb{P}, \mathcal{F} = 2^{2^{\mathcal{X}}})$ such that, for $X \subseteq \mathcal{X}$,

$$(3.1) \qquad\qquad \mathbb{P}[\{X\}] = \frac{\det L_X}{\det(\mathbf{1} + L)}.$$

Here L_X means pick out those rows and columns that correspond to X, this giving a $|X| \times |X|$ matrix. Also, $\mathbf{1}$ is the identity matrix of appropriate size. For sets in $\mathcal{F}$ which are not singletons, the measure $\mathbb{P}$ is defined by the additivity property.

This is only a well defined probability if the expression (3.1) is always positive, for example if L is positive definite. The fact that the probabilities sum up to one is guaranteed by the following well known formula, the Fredholm expansion of a determinant.

LEMMA 3.1. — *Let M be a matrix whose rows and columns are indexed by the finite set $\mathcal{X}$. Then*

$$\det(\mathbf{1} + M) = \sum_{X \subseteq \mathcal{X}} \det M_X,$$

with the understanding that the determinant of the empty matrix is 1.

We do not show this, it can for example be done by induction over the size of the matrix. To convince the reader of the validity and triviality of the above lemma, let us see what happens with a 2×2 matrix.

$$\det \begin{bmatrix} 1+a & b \\ c & 1+d \end{bmatrix} = \det \begin{bmatrix} 1 & b \\ 0 & 1+d \end{bmatrix} + \det \begin{bmatrix} a & b \\ c & 1+d \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} + \det \begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix} + \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= 1 + \det \begin{bmatrix} d \end{bmatrix} + \det \begin{bmatrix} a \end{bmatrix} + \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The first equality comes from the fact that the determinant is linear in the first and second column. The second equality comes from expanding along rows.

THEOREM 3.2. — Let $K = L(1+L)^{-1}$. Then for all $X \in 2^{\mathcal{X}}$,

$$\mathbb{P}[E_{\mathcal{X}}(X)] = \det K_X.$$

The matrix K is frequently called the correlation kernel of $\mathbb{P}$.

Proof. — Let $\mathbf{1}_{(X)}$ be the identity matrix with the ones corresponding to elements in $\bar{X} := \mathcal{X} \setminus X$ set to zero.

$$\mathbb{P}[E_{\mathcal{X}}(X)] = \sum_{X \subseteq Y \subseteq \mathcal{X}} \mathbb{P}[\{Y\}]$$

$$= \det(1+L)^{-1} \sum_{X \subseteq Y \subseteq \mathcal{X}} \det L_Y$$

$$= \det(\mathbf{1}_{(\bar{X})} + L)(1+L)^{-1}$$

$$= \det(1 + L - \mathbf{1}_{(X)})(1+L)^{-1}$$

$$= \det(1 - \mathbf{1}_{(X)}(1+L)^{-1})$$

$$= \det[1 - (1+L)^{-1}]_X$$

$$= \det[L(1+L)^{-1}]_X.$$

The third equality is due to a variant of Lemma 3.1. □

3.4. Projection on a subspace

Given an L-ensemble on $\mathcal{X}$, let us take an arbitrary fixed subset $\mathcal{N} \subset \mathcal{X}$. Define its conjugate $\bar{\mathcal{N}} := \mathcal{X} \setminus \mathcal{N}$. We want to study a certain projection $\mathbb{P}^*$

of $\mathbb{P}$ to $2^{2^{\mathcal{N}}}$. This shall give us a smaller point process $(\Omega^* = 2^{\mathcal{N}}, \mathbb{P}^*, \mathcal{F}^* = 2^{2^{\mathcal{N}}})$, specified by

$$\mathbb{P}^*[\{D\}] = \frac{\mathbb{P}[\{D \cup \bar{\mathcal{N}}\}]}{\mathbb{P}[E_{\mathcal{X}}(\bar{\mathcal{N}})]}.$$

Again $\mathbb{P}^*$ is defined by additivity for events that aren't singletons. This should be thought of as a conditional probability. Compute for example

$$\begin{aligned}
\mathbb{P}^*[E_{\mathcal{N}}(D)] &= \mathbb{P}^*[\cup_{D \subseteq F \subseteq \mathcal{N}} \{F\}] \\
&= \sum_{D \subseteq F \subseteq \mathcal{N}} \mathbb{P}^*[\{F\}] \\
&= \sum_{D \subseteq F \subseteq \mathcal{N}} \frac{\mathbb{P}[\{F \cup \bar{\mathcal{N}}\}]}{\mathbb{P}[E_{\mathcal{X}}(\bar{\mathcal{N}})]} \\
&= \frac{\mathbb{P}[E_{\mathcal{X}}(D \cup \bar{\mathcal{N}})]}{\mathbb{P}[E_{\mathcal{X}}(\bar{\mathcal{N}})]} \\
&= \frac{\mathbb{P}[E_{\mathcal{X}}(D) \cap E_{\mathcal{X}}(\bar{\mathcal{N}})]}{\mathbb{P}[E_{\mathcal{X}}(\bar{\mathcal{N}})]}.
\end{aligned}$$

THEOREM 3.3. — *Let* $K^* = \mathbf{1}_{(\mathcal{N})} - (\mathbf{1}_{(\mathcal{N})} + L)^{-1}|_{\mathcal{N}}$. *Then for all* $X \in \mathcal{F}^*$,

$$\mathbb{P}^*[E_{\mathcal{N}}(X)] = \det K_X^*.$$

Here, $|_{\mathcal{N}}$ means pick out those rows and columns that correspond to $\mathcal{N}$. For proof see [10].

3.5. Eynard-Mehta Theorem

Again let us specialise. We are interested in studying a point process on the space $\mathcal{N} = \mathcal{X}^{(0)} \sqcup \mathcal{X}^{(1)} \sqcup \cdots \sqcup \mathcal{X}^{(N)}$ which is the disjoint union of N finite sets $\mathcal{X}^{(n)}$ for $n = 1, 2, \ldots, N$.

A sample $x \in \mathcal{N}$ can be written

$$\bar{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(N)})$$

where $x^{(n)} \in 2^{\mathcal{X}^{(n)}}$. Fix an integer p and the following functions:

$$\begin{aligned}
&\phi_k : \mathcal{X}^{(0)} \to \mathbb{R}, \\
(3.2) \qquad &W_n : \mathcal{X}^{(n)} \times \mathcal{X}^{(n+1)} \to \mathbb{R}, \text{ and} \\
&\psi_k : \mathcal{X}^{(N)} \to \mathbb{R}
\end{aligned}$$

for $n = 0, \ldots, N-1$ and $k = 1, \ldots, p$. The measure we are interested in has the form

$$(3.3) \quad \mathbb{P}^*[\{\bar{x}\}] = Z^{-1} \det[\phi_k(x_l^{(0)})]_{k,l=1}^p \det[W_0(x_k^{(0)}, x_l^{(1)})]_{k,l=1}^p \times$$
$$\cdots \times \det[W_{N-1}(x_k^{(N-1)}, x_l^{(N)})]_{k,l=1}^p \det[\psi_k(x_l^{(N)})]_{k,l=1}^p$$

if $|x^{(1)}| = |x^{(2)}| = \cdots = |x^{(N)}| = p$ and $\mathbb{P}^*[\{\bar{x}\}] = 0$ otherwise. Here $x_l^{(n)}$ is the l:th element of the set $x^{(n)}$ for $n = 1, \ldots, N$ and $l = 1, \ldots, p$. For that to be well defined one needs a total ordering on $\mathcal{X}^{(n)}$, but which one we use is not important. Note that the measure $\mathbb{P}^* : 2^{2^{\mathcal{N}}} \to \mathbb{R}$ is defined by (3.3) for singleton sets and by additivity for all other sets in $2^{2^{\mathcal{N}}}$. Measures of this form turn up everywhere in random matrix theory and related combinatorial models.

The idea now is to extend the space on which the point process lives in such a way that the new bigger process admits an L-ensemble representation. Let $\bar{\mathcal{N}} = \{1, 2, \ldots, p\}$ and let $\mathcal{X} = \bar{\mathcal{N}} \sqcup \mathcal{N}$. The measure $\mathbb{P}^*$ can then be expressed as

$$\mathbb{P}^*[\{\bar{x}\}] = \frac{\mathbb{P}[\{\tilde{x} \sqcup \bar{\mathcal{N}}\}]}{\mathbb{P}[E_\mathcal{X}(\bar{\mathcal{N}})]}$$

where $\mathbb{P}$ is the measure defined by (3.1) where

$$(3.4) \qquad L = \begin{bmatrix} 0 & \Phi & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -W_0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -W_1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & -W_{N-1} \\ \Psi & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Here Φ, $W_1, \ldots, W_{N-1}$ and Ψ are certain blocks and 0 means the zero matrix of appropriate dimension. The minus signs are for convenience later. Recall that the L-matrix should be of size $|\mathcal{X}| \times |\mathcal{X}|$ and that its rows and columns are indexed by elements of $\mathcal{X} = \bar{\mathcal{N}} \sqcup \mathcal{X}^{(1)} \sqcup \mathcal{X}^{(2)} \sqcup \cdots \sqcup \mathcal{X}^{(N)}$. The determinants of the various blocks in (3.4) will be exactly the determinants that occur in (3.3).

Here, Φ and Ψ are matrices of dimension $p \times |\mathcal{X}^{(0)}|$ and $|\mathcal{X}^{(N)}| \times p$ respectively defined by

$$[\Phi]_{n,x} = \phi_n(x) \quad \text{for } n \in \bar{\mathcal{N}} \text{ and } x \in \mathcal{X}^{(0)},$$
$$[\Psi]_{x,n} = \psi_n(x) \quad \text{for } x \in \mathcal{X}^{(N)} \text{ and } n \in \bar{\mathcal{N}}.$$

The matrices W_n for $n = 1, \ldots, N$ are of size $|\mathcal{X}^{(n)}| \times |\mathcal{X}^{(n+1)}|$ and defined by (3.2). Let

(3.5) $$W_{[n,m)} = \begin{cases} W_n \cdots W_{m-1}, & n < m, \\ 0, & n \geqslant m. \end{cases}$$

THEOREM 3.4 (Eynard-Mehta Theorem). — *Assume*

(3.6) $$M := \Phi W_0 \cdots W_{N-1} \Psi$$

is invertible. Then there exists a correlation kernel K^* for the measure $\mathbb{P}^*$, that is,

$$\mathbb{P}^*[\, E_\mathcal{N}(X)\,] = \det K_X^*.$$

This matrix can be written in block form

$$K^* = \begin{bmatrix} K_{0,0}^* & \cdots & K_{0,N}^* \\ \vdots & & \\ K_{N,0}^* & \cdots & K_{N,N}^* \end{bmatrix}$$

where

(3.7) $$K_{n,m}^* = W_{[n,N)} \Psi M^{-1} \Phi W_{[0,m)} - W_{[n,m)}.$$

Note that the block $K_{n,m}^*$ is of size $|\mathcal{X}^{(n)}| \times |\mathcal{X}^{(m)}|$.

Proof. — We shall use the following matrix identity.

(3.8) $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} -M^{-1} & M^{-1}BD^{-1} \\ D^{-1}CM^{-1} & D^{-1} - D^{-1}CM^{-1}BD^{-1} \end{bmatrix}.$$

Here, A and B must be square blocks and $M = BD^{-1}C - A$. This is easy to verify by explicit computation.

Now according to Theorem 3.3 we need to invert $1_{(\mathcal{N})} + L$ which can be decomposed as the left hand side of (3.8) with $A = 0$, $B = [\Phi, 0, 0, \ldots]$,

(3.9)
$$D^{-1} = \begin{bmatrix} \mathbf{1} & -W_0 & 0 & \cdots & 0 \\ 0 & \mathbf{1} & -W_1 & \cdots & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \mathbf{1} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{1} & W_{[0,1)} & W_{[0,2)} & \cdots & W_{[0,N)} \\ 0 & \mathbf{1} & W_{[1,2)} & \cdots & W_{[1,N)} \\ 0 & 0 & \mathbf{1} & \cdots & W_{[2,N)} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \mathbf{1} \end{bmatrix}$$

and C appropriately chosen.

Thus

$$
BD^{-1} = [\Phi \quad \Phi W_{[0,1)} \quad \cdots \quad \Phi W_{[0,N)}], \qquad D^{-1}C = \begin{bmatrix} W_{[0,N)}\Psi \\ W_{[1,N)}\Psi \\ \vdots \\ \Psi \end{bmatrix},
$$

and M is as given by (3.6). Applying (3.8) to the formula in Theorem 3.3 gives

$$
K^* = \mathbf{1} - D^{-1} + D^{-1}CM^{-1}BD^{-1}.
$$

Inserting the various ingredients above into this formula proves Theorem 3.4 .   □

4.   Tips and Tricks

To compute the kernel in the main theorem, the following additional ideas are needed. None of these are new but for the instruction of the reader they are summarised here.

4.1. Continuous state space

In the version of the Eynard-Mehta theorem above, Theorem 3.4, the state spaces $\mathcal{X}^{(n)}$, for $n = 1, \ldots, N$, are finite. In the literature a version with $\mathcal{X}^{(n)} = \mathbb{R}$ for all n is more common. We shall now expound on the relationship between these two versions of the same useful theorem.

Again we are faced with analysing a measure on the form (3.3) but now $\mathbb{P}^*[\{x^*\}]$ is the probability density of configuration x^*,

$$
\begin{aligned}
\phi_n &: \mathbb{R} \to \mathbb{R}, & \text{for } n &= 1, \ldots, p, \\
W_n &: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, & \text{for } n &= 1, \ldots, N, \\
\psi_n &: \mathbb{R} \to \mathbb{R}, & \text{for } n &= 1, \ldots, p.
\end{aligned}
$$

The state space is now

$$(4.1) \qquad\qquad \mathbb{R} \sqcup \mathbb{R} \sqcup \cdots \sqcup \mathbb{R} = N \times \mathbb{R}.$$

Pick some discretisation $\mathcal{M}$ of the real line, *i.e.* some sequence $\mathcal{M}_1$, $\mathcal{M}_2, \ldots$ of $|\mathcal{M}|$ real numbers. Restricting the measure in (3.3) — with state space given by (4.1) — to the state space

$$\mathcal{M} \sqcup \mathcal{M} \sqcup \cdots \sqcup \mathcal{M} = N \times \mathcal{M}$$

gives a measure on a discrete set of exactly the type to which Theorem 3.4 applies. Now all the blocks in for example (3.9) are $|\mathcal{M}| \times |\mathcal{M}|$ and thus D is a matrix of size $(N|\mathcal{M}|) \times (N|\mathcal{M}|)$. The idea is of course to take the limit $|\mathcal{M}| \to \infty$.

The correlation kernel (3.7) is computed through suitable matrix multiplications and inversions. Consider for example the matrix multiplication of W_1 and W_2.

$$\frac{1}{|\mathcal{M}|}[W_1 W_2]_{x,z} = \frac{1}{|\mathcal{M}|} \sum_{y \in \mathcal{M}} [W_1]_{x,y}[W_2]_{y,z} \to \int_{\mathbb{R}} W_1(x,y)W_2(y,z)\,dy,$$

$$|\mathcal{M}| \longrightarrow \infty.$$

The constant $|\mathcal{M}|^{-1}$ can be absorbed into the normalisation constant Z. Thus we see that all the matrix multiplications in the expression (3.7) turn into convolutions of the corresponding functions in the continuous setting. In later sections we will blur the line between discrete and continuous by sometimes using the notation of matrix multiplication for convolutions. It is understood that one needs to check convergence and integrability when one goes from the discrete to the continuous. That offers no problem in our examples so nothing further will be said on that score.

4.2. Unequal number of particles on each level

We shall illustrate this by the example of the GUE Minor process. Consider a GUE random matrix M of size $N \times N$. Let the eigenvalues of the $n \times n$ minor, that is $[M_{ij}]_{i,j=1}^n$, be denoted $\lambda_1^{(n)} > \cdots > \lambda_n^{(n)}$. Then these vectors $\lambda^{(1)}, \ldots, \lambda^{(N)}$ can be seen as random variables.

For this process, the probability measure for all the variables $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ is absolutely continuous with respect to the Lebesgue measure, thus it has a *probability density function* (p.d.f.). It can be written [3, 25, 18] as

$$p(\bar{\lambda}) = \frac{1}{C} \mathbf{1}\{\lambda^{(1)} \prec \lambda^{(2)}\} \cdots \mathbf{1}\{\lambda^{(N-1)} \prec \lambda^{(N)}\} \, \Delta(\lambda^{(N)}) \prod_{n=1}^{N} e^{-(\lambda_n^{(N)})^2}.$$

Here, $\lambda^{(n)} \prec \lambda^{(n+1)}$ means $\lambda^{(n)}$ and $\lambda^{(n+1)}$ interlace, i.e. $\lambda_1^{(n+1)} \geqslant \lambda_1^{(n)} \geqslant \lambda_2^{(n+1)} \geqslant \cdots \geqslant \lambda_n^{(n+1)} \geqslant \lambda_n^{(n)} \geqslant \lambda_{n+1}^{(n+1)}$. We use the increasingly common notation that Δ denotes the Vandermonde determinant. It turns out that it is practical to introduce fictitious (or virtual) variables $\lambda_1^{(0)} = \lambda_2^{(1)} = \lambda_3^{(2)} = \cdots = \lambda_N^{(N-1)} = -\infty$. Then the interlacing condition can be written in terms

of determinants [39] using the Heaviside function $H(x) = \mathbf{1}\{x \geqslant 0\}$ and the above p.d.f. becomes

$$(4.2) \quad p(\bar{\lambda}) = \frac{1}{C} \det[H(\lambda_i^{(1)} - \lambda_j^{(0)})]_{i,j=1}^1 \det[H(\lambda_i^{(2)} - \lambda_j^{(1)})]_{i,j=1}^2 \cdots$$

$$\cdots \det[H(\lambda_i^{(N)} - \lambda_j^{(N-1)})]_{i,j=1}^N \Delta(\lambda^{(N)}) \prod_{n=1}^N e^{-(\lambda_n^{(N)})^2}.$$

The reader must agree that this vaguely resembles (3.3) except that the dimension of the matrices change. The first is a 1×1 determinant and the last of size $N \times N$ for example. Notice too that the last column of all these matrices is identically one because of our choice of fictitious particles above.

The way to deal with this, first discovered in [7], is to form an L-matrix similar to the one in (3.4) but which looks like this.

$$(4.3) \qquad L = \begin{bmatrix} 0 & \Phi & 0 & 0 & \cdots & 0 & 0 \\ E_0 & 0 & -W_0 & 0 & \cdots & 0 & 0 \\ E_1 & 0 & 0 & -W_1 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ E_{N-1} & 0 & 0 & 0 & \cdots & 0 & -W_{N-1} \end{bmatrix}.$$

Here, in the example of the GUE Minor process, $[W_n]_{x,y} = H(y - x)$ for $n = 0, 1, \ldots, N-1$, and $x, y \in \mathcal{M}$. For the matrices in (4.2) the last column — which is identically one — is moved out to the first column of blocks. Thus a sequence of $\mathcal{M} \times N$ matrices $E_0, \ldots, E_{N-1}$ are produced such that, for $m = 1, \ldots, N$,

$$[E_n]_{x,m} = \begin{cases} 1, & n+1 = m, \\ 0, & n+1 \neq m. \end{cases}$$

By cranking this machinery it is possible to compute the correlation kernel for the GUE Minor process. This is done in [7, 18].

4.3. Column operations on the kernel

Recall the expression for the kernel in (3.7). It is sometimes favourable to perform some column operations on the matrices $W_{[n,N)}\Psi$ for $n = 0, \ldots, N - 1$. Doing *column operations* means multiplying from the right with an upper triangular or lower triangular matrix, say R_m, which must be

invertible. Typically, but not necessarily, it will have ones on the diagonal and a single off-diagonal entry. The kernel (3.7) then takes the form

$$K_{n,m}^* = W_{[n,N)}\Psi R_n(MR_n)^{-1}\Phi W_{[0,m)} - W_{[n,m)}$$

for $n, m \in \{0, \ldots, N-1\}$.

5. Markov property along space-like paths

It is a sad fact of life that no transition density for the Warren process is known explicitly. We do however know certain marginals. To fix notation let $\bar{x} = (x^{(1)}, \ldots, x^{(n)})$ where $x^{(k)} \in \mathbb{R}^k$. Let $I^{(n)}(\bar{x}) = 1$ if the interlacing $x^{(n)} \succ \cdots \succ x^{(1)}$ holds and 0 otherwise. Let $P_t^{(n)}$ be the transition density for Warren's process which we know exists since it is a well defined Markov process. In this notation, [39, Proposition 6] can be restated as follows.

PROPOSITION 5.1. — For fixed n, $x^{(n)}$ and $\bar{y} = (y^{(1)}, \ldots, y^{(n)})$,

$$\int \frac{I^{(n)}(\bar{x})}{\Delta(x^{(n)})} P_t^{(n)}(\bar{x}, \bar{y})\, dx^{(1)} \cdots dx^{(n-1)} = \frac{I^{(n)}(\bar{y})}{\Delta(y^{(n)})} p_t^{(n)}(x^{(n)}, y^{(n)})$$

where

$$p_t^{(n)}(x, y) = \frac{\Delta(y)}{\Delta(x)} \det \left[e^{-(x_i - y_j)^2/t} \right]_{i,j=1}^n.$$

Also, by the characterisation after (30) in Warren's paper, it is clear that

PROPOSITION 5.2. — For fixed n, $x^{(1)}, \ldots, x^{(n-1)}$, $y^{(1)}, \ldots, y^{(n-1)}$,

$$\int P_t^{(n)}(x^{(1)}, \ldots, x^{(n)}; y^{(1)}, \ldots, y^{(n)})\, dy^{(n)}$$
$$= P_t^{(n-1)}(x^{(1)}, \ldots, x^{(n-1)}; y^{(1)}, \ldots, y^{(n-1)}).$$

Notice that there is no dependence on $x^{(n)}$ in the right hand side, *i.e.* the evolution of the particles on levels $1, \ldots, n-1$ does not feel what happens on level n.

We shall compute the eigenvalue measure along a particular space-like path and the reader will see how to generalise this. Suppose we want to look at the path (n, t_1), (n, t_2), $(n-1, t_2)$, $(n-2, t_2)$, $(n-2, t_3)$ for some fixed n and $0 < t_1 < t_2 < t_3$. The density of the event that the Warren process, started at the origin, takes values $\bar{x}$, $\bar{y}$ and $\bar{z}$ respectively at times t_1, t_2 and t_3 respectively is by [39]

$$\Delta^2(x^{(n)}) e^{-\sum_{i=1}^n (x_i^{(n)})^2} \frac{I^{(n)}(\bar{x})}{\Delta(x^{(n)})} P_t^{(n)}(\bar{x}, \bar{y}) P_{t'}^{(n)}(\bar{y}, \bar{z})$$

where $t := t_2 - t_1$ and $t' := t_3 - t_2$. To find the distribution on the afore-mentioned path we need to integrate out $x^{(1)}, \ldots, x^{(n-1)}$, $y^{(1)}, \ldots, y^{(n-3)}$, $z^{(n-1)}$ and $z^{(n)}$.

We start by integrating out the unwanted x-variables which can be done by applying Proposition 5.1 which gives

$$(5.1) \qquad \Delta^2(x^{(n)}) e^{-\sum (x^{(n)})^2} p_t^{(n)}(x^{(n)}, y^{(n)}) \frac{I^{(n)}(\bar{y})}{\Delta(y^{(n)})} P_{t'}^{(n)}(\bar{y}, \bar{z}).$$

Observing that

$$I^{(n)}(\bar{y}) = \mathbf{1}\{y^{(n)} \succ y^{(n-1)} \succ y^{(n-2)}\} I^{(n-2)}(y^{(1)}, \ldots, y^{(n-2)})$$

and applying Proposition 5.2 twice we see that

$$\int I(\bar{y}) P_{t'}^{(n)}(\bar{y}, \bar{z}) \, dz^{(n)} dz^{(n-1)}$$
$$= \mathbf{1}\{y^{(n)} \succ y^{(n-1)} \succ y^{(n-2)}\} P_{t'}^{(n-2)}(y^{(1)}, \ldots, y^{(n-2)}; z^{(1)}, \ldots, z^{(n-2)}).$$

We insert that into (5.1) integrated and then apply Proposition 5.1 to integrate out the unwanted y-variables to get

$$(5.2)$$
$$\Delta(x^{(n)}) e^{-\sum (x^{(n)})^2} \frac{\Delta(x^{(n)})}{\Delta(y^{(n)})} p_t^{(n)}(x^{(n)}, y^{(n)}) \mathbf{1}\{y^{(n)} \succ y^{(n-1)} \succ y^{(n-2)}\} \times$$
$$\times \frac{\Delta(y^{(n-2)})}{\Delta(z^{(n-2)})} p_{t'}^{(n-2)}(y^{(n-2)}, z^{(n-2)}) \frac{I^{(n-2)}(z^{(1)}, \ldots, z^{(n-2)})}{\Delta(z^{(n-2)})}.$$

As was noted before (4.2), with $H(x) = \mathbf{1}\{x > 0\}$,

$$\mathbf{1}\{y^{(n)} \succ y^{(n-1)}\} = \det[H(y_i^{(n)} - y_j^{(n-1)})]_{i,j=1}^n$$

if you adopt the convention that $y_n^{(n-1)} = -\infty$. Then (5.2) can be written as a product of determinants.

$$(5.3) \quad \Delta(x^{(n)}) e^{-\sum (x^{(n)})^2} \det \left[e^{-(x_i^{(n)} - y_j^{(n)})^2/t} \right]_{i,j=1}^n \times$$
$$\times \det \left[H(y_i^{(n)} - y_j^{(n-1)}) \right]_{i,j=1}^n \det \left[H(y_i^{(n-1)} - y_j^{(n-2)}) \right]_{i,j=1}^{n-1} \times$$
$$\times \det \left[e^{-(y_i^{(n-2)} - z_j^{(n-2)})^2/t'} \right]_{i,j=1}^{n-2} \frac{I^{(n-2)}(z^{(1)}, \ldots, z^{(n-2)})}{\Delta(z^{(n-2)})}.$$

The same idea can be applied to any other space-like path.

An argument for the corresponding statement for the Dyson Brownian minor process is given in [17, Section 4].

6. Definitions and computations

Given the theory presented in the last two sections, computing the kernel is nothing but a long tedious computation. It was hard to write, hopefully it isn't too hard to read. The computation for the Warren process and the Dyson BM process can be done at the same time with judicious choice of notation. Let

(6.1) $$H^n(x) := \begin{cases} (n-1)!^{-1}x^{n-1}\mathbf{1}\{x \geqslant 0\}, & n = 1,\, 2,\, \ldots, \\ \delta(x), & n = 0. \end{cases}$$

be the nth anti-derivative of the Dirac delta function and $H := H^1$ be the Heaviside function.

6.1. Dyson BM

Define the normalised Hermite polynomials,

$$h_n^*(x) = [\sqrt{\pi}n!2^n]^{-1/2}(-1)^n(w^*(x))^{-1}D^n w^*(x)$$

which are orthonormal with respect to the weight

$$w^*(x) = e^{-x^2}.$$

The transition density of the Ornstein-Uhlenbeck process is the well known expression

(6.2) $$p_t^*(x,y) = \frac{\exp(\frac{-(y-q_t^* x)^2}{1-q_t^{*,2}})}{\sqrt{\pi(1-q_t^{*,2})}}$$

where

$$q_t^* = e^{-t}.$$

While we are at it, define $r_t^* := q_t^*$ and $\sigma^*(t) = 1/\sqrt{2}$ for $t > 0$, and set $q_t^{*,n} = (q_t^*)^n$.

6.2. Warren process

For $t > 0$, let

$$h_n^{(t)}(x) = [\sqrt{\pi}n!2^n t^{-n}]^{-1/2}(-1)^n(w^{(t)}(x))^{-1}D^n w^{(t)}(x).$$

These are orthogonal with respect to the weight

$$w^{(t)}(x) = e^{-x^2/t}$$

and related to the Hermite polynomials above by $h^{(t)}(x) = h^*(x/\sqrt{t})$.

The transition density of Brownian motion with variance $t/2$ is

(6.3)
$$p_t(x, y) = \frac{1}{\sqrt{\pi t}} e^{-(y-x)^2/t}.$$

Define

$$q_s^{(t)} = \sqrt{\frac{t}{s+t}}.$$

We are going to set $r_t \equiv 1$ and $\sigma(t) = \sqrt{t/2}$. Also, as notational convenience let $q_s^{(t),n} = (q_s^{(t)})^n$.

We need the following convolutions in our computations later.

LEMMA 6.1. — For $n = 0, 1, 2, \ldots$, and $t > 0$,

(6.4)
$$\int_{\mathbb{R}} h_n^{(t)}(x) w^{(t)}(x) p_s(x, y) \, dx = q_s^{(t),n} h_n^{(t+s)}(y) w^{(t+s)}(y),$$

(6.5)
$$\int_{\mathbb{R}} h_{n+1}^{(t)}(x) w^{(t)}(x) H(x - y) \, dx = \sigma(t)(n+1)^{-1/2} h_n^{(t)}(y) w^{(t)}(y),$$

(6.6)
$$\int_{\mathbb{R}} H^n(x - y) H(y - z) \, dy = H^{n+1}(x - z)$$

(6.7)
$$\int_{\mathbb{R}} p_t(x, y) p_s(y, z) \, dy = p_{t+s}(x, z)$$

(6.8)
$$\int_{\mathbb{R}} p_t(x, y) H^n(y - z) \, dy = r_t^n \int_{\mathbb{R}} H^n(x - y) p_t(y, z) \, dy$$

(6.9)
$$\int_{\mathbb{R}} h_n^{(t)}(x) h_m^{(t)}(x) w^{(t)}(x) = \sqrt{2} \sigma(t) \delta_{nm}$$

(6.10)
$$p_t(x + y, z) = p_t(x, z - r_t y),$$

(6.11)
$$q_{t_2 - t_1}^{(t_1)} q_{t_3 - t_2}^{(t_2)} = q_{t_3 - t_1}^{(t_1)}$$

(6.12)
$$r_t r_s = r_{t+s}$$

(6.13)
$$\frac{q_{t-s}^{(s)}}{\sigma(s)} = \frac{r_{t-s}}{\sigma(t)}.$$

All of this is also true for the stared functions. The coefficient of x^n in $h_n^{(t)}$ is

$$a_n := \frac{1}{\sigma^n(t)\sqrt{n!\sqrt{\pi}}}$$

and that is true for h_n^* with σ replaced by σ^*.

These are proved by explicit elementary computation. An important point here is that replacing h, q, r, w, p and σ with the stared versions these equations still hold. By this intelligent choice of notation we can do

the computation concerning the Ornstein-Uhlenbeck and the Warren process at the same time. Furthermore,

LEMMA 6.2.

$$\int_{\mathbb{R}} H^n(x-y)p_{t-s}(y,z)\,dy$$

$$= \sum_{k=0}^{n-1} \frac{h_k^{(s)}(x)(\sigma(t))^k q_{t-s}^{(s),k}}{r_{t-s}^n \sigma(s)\sqrt{2k!}\pi^{\frac{1}{4}}} \int_{\mathbb{R}} w^{(t)}(y)H^{n-k}(y-z)\,dy$$

$$+ \frac{(\sigma(t))^n}{\sqrt{2}\sigma(s)r_{t-s}^n} \sum_{k=n}^{\infty} \sqrt{\frac{(k-n)!}{k!}} q_{t-s}^{(s),k} h_k^{(s)}(x) h_{k-n}^{(t)}(z)w^{(t)}(z).$$

Proof. — By orthogonality

$$f(x,z) = \int H^n(x-y)p_{t-s}(y,z)\,dy$$

can be written as

$$f(x,z) = \sum_{k=0}^{\infty} c_k(z)h_k^{(s)}(x)$$

for suitable coefficients

(6.14)

$$(\sqrt{2}\sigma(s))c_k(z) = \int_{\mathbb{R}} f(x,z)h_k^{(s)}(x)w^{(s)}(x)\,dx$$

(6.15)
$$= \iint_{\mathbb{R}\times\mathbb{R}} H^n(x-y)p_{t-s}(y,z)h_k^{(s)}(x)w^{(s)}(x)\,dx\,dy.$$

Let us start with the case $k \geqslant n$. Apply (6.8).

(6.16)
$$= \frac{1}{r_{t-s}^n} \iint_{\mathbb{R}\times\mathbb{R}} h_k^{(s)}(x)w^{(s)}(x)p_{t-s}(x,y)H^n(y-z)\,dx\,dy.$$

Apply (6.4).

(6.17)
$$= \frac{q_{t-s}^{(s),k}}{r_{t-s}^n} \int_{\mathbb{R}} h_k^{(t)}(y)w^{(t)}(y)H^n(y-z)\,dy.$$

Apply (6.5) and (6.6) n times to get

(6.18)
$$= \frac{q_{t-s}^{(s),k}}{r_{t-s}^n}(\sigma(t))^n \sqrt{\frac{(k-n)!}{k!}} h_{k-n}^{(t)}(z)w^{(t)}(z).$$

Now suppose instead that $k < n$. Then everything up to (6.17) works the same. Apply (6.5) k times.

$$(6.19) \qquad = \frac{q_{t-s}^{(s),k}}{r_{t-s}^n} \frac{(\sigma(t))^k}{\sqrt{k!}} \int_{\mathbb{R}} h_0^{(t)}(y) w^{(t)}(y) H^{n-k}(y-z) \, dy.$$

Remember that $h_0(y) \equiv \pi^{-\frac{1}{4}}$.

$$(6.20) \qquad = \frac{q_{t-s}^{(s),k}}{r_{t-s}^n} \frac{(\sigma(t))^k}{\sqrt{k!}\pi^{\frac{1}{4}}} \int_{\mathbb{R}} w^{(t)}(y) H^{n-k}(y-z) \, dy.$$

$\square$

6.3. Integral representations

With the normalisations above the classical integral representations for the Hermite polynomials are

$$(6.21) \qquad h_n^{(t)}(x) = \frac{\pi^{\frac{1}{4}} e^{x^2/t} (2t)^{n/2}}{i\pi\sqrt{n!}} \int_{\Gamma} v^n e^{v^2 - 2vx/\sqrt{t}} \, dv$$

and

$$(6.22) \qquad h_n^{(t)}(x) = \frac{(t/2)^{n/2}\sqrt{n!}}{\pi^{\frac{1}{4}}} \frac{1}{2\pi i} \int_{\gamma} u^{-n-1} e^{-u^2 + 2ux/\sqrt{t}} \, du$$

with contours of integration as in Figure 1.1. The starred Hermite polynomials are given, for $n = 0, 1, \ldots$, by $h_n^* \equiv h_n^{(1)}$.

As a sort of generalisation of (6.21) it can be shown [34] that for $n = 1, 2, \ldots$,

$$\frac{1}{\pi i} \int_{\Gamma} v^{-n} e^{v^2 - 2vx/\sqrt{t}} \, dv = \frac{2^n}{\sqrt{\pi} t^{n/2}} \int_{\mathbb{R}} H^n(y-x) e^{-y^2/t} \, dy.$$

7. The Kernel

We will now introduce some notation, and hopefully Figure 7.1 will clarify the situation. Pick N times and levels $(t_0, n_0), \ldots, (t_{N-1}, n_{N-1})$ following a space-like path. This means that $0 < t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{N-1}$ and $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_{N-1}$. Without loss of generality we can take $n_{N-1} = 1$ and $n_m - n_{m+1} \in \{0, 1\}$ for all $m = 0, \ldots, N-2$. That is, we end at level 1 and only drop one level at a time. For the sake of notation let $n_N = 0$ and $t_N = t_{N-1}$. Denote by $x^{(k)} = (x_1^{(k)} \geqslant x_2^{(k)} \geqslant \cdots \geqslant x_{n_k}^{(k)})$ the n_k
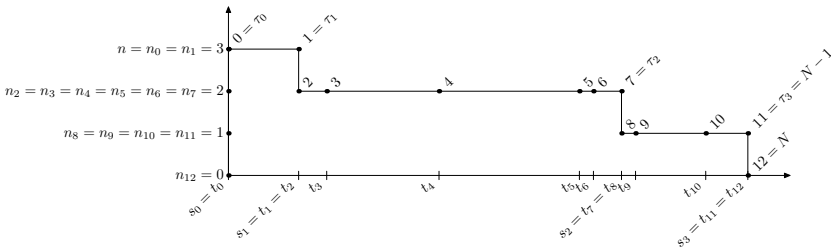
Figure 7.1. Times and levels must follow a space-like path. This means that this curve must not take steps upward.

eigenvalues at time t_k and level n_k. We shall say that $m \in$ Space if the mth step is a down step, i.e. $n_m = n_{m+1} + 1$. For $k = 1, \ldots, n_0$, let τ_k and s_k be the position and time, respectively, of the kth down step. Thus $\tau_k \in$ Space by definition and $n_{\tau_k} = n + 1 - k$. The time of the kth down step is $s_k = t_{\tau_k} = s_{n_0 + 1 - n_{\tau_k}}$. Also let $\tau_0 = 0$ and $s_0 = t_0$. If the mth step is a time step, i.e. $n_m = n_{m+1}$, then we shall write $m \in$ Time. That's a lot of notation, hopefully Figure 7.1 should make this clearer.

Let

$$\phi_k(x) = h_{k-1}^{(t_0)}(x) w^{(t_0)}(x) \qquad \text{for } k = 1, \ldots, n_0$$

$$W_m(x, y) = \begin{cases} p_{t_{m+1} - t_m}(x, y) & m \in \text{Time} \\ H(x - y) & m \in \text{Space} \end{cases} \qquad \text{for } m = 0, \ldots, N - 1.$$

A full configuration of eigenvalues is the N-tuple

$$\bar{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(N-1)})$$

where $x^{(m)} \in \mathbb{R}^{n_m}$ for $m = 0, \ldots, N - 1$. We adopt the notation that $x_{n_m+1}^{(m+1)} = u$ for some large negative real number u if $m \in$ Space. In particular $x_1^N = u$ since, by definition $N - 1 \in$ Space. Those are the positions where we step down a level, that is, we lose an eigenvalue. One way to think of this is that that particle jumps away to some position u which is close to $-\infty$. The weight or probability density of configuration $\bar{x}$ is then, by the Markov property along space-like paths discussed in Section 5, given by the following product

$$(7.1) \qquad p(\bar{x}) = Z^{-1} \det[\phi_k(x_l^{(0)})]_{k,l=1}^{n_0} \prod_{m=0}^{N-1} \det[W_m(x_k^{(m)}, x_l^{(m+1)})]_{k,l=1}^{n_m},$$

where we adopt the notation that $x_{n_m+1}^{(m+1)} = u$ for some large negative real number u if $n_m - n_{m+1} = 1$, for $m = 1, \ldots, N$. Also $x_1^N = u$.

Let the magic begin. Define the block matrix

$$
L = \begin{bmatrix}
0 & \Phi & 0 & 0 & \cdots & 0 & 0 \\
F_0 & 0 & -W_0 & 0 & \cdots & 0 & 0 \\
F_1 & 0 & 0 & -W_1 & \cdots & 0 & 0 \\
\vdots & & & & & & \\
F_{N-2} & 0 & 0 & 0 & \cdots & -W_{N-2} & 0 \\
F_{N-1} & 0 & 0 & 0 & \cdots & 0 & -W_{N-1} \\
F_N & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
$$

The F_N block in the above block matrix has zero rows but let us keep this notation. The $(W_n)_{n=0}^{N-1}$ are defined above. For $l = 1, \ldots, n_0$,

(7.2) $$[\Phi]_{l,x} = \phi_{n_0+1-l}(x) = h_{n_0-l}^{(t_0)}(x) w^{(t_0)}(x),$$

(7.3) $$[F_k]_{x,l} = \begin{cases} H(x-u), & k = \tau_{n_0-l+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The measure in (7.1), being similar to that in (3.3), can then be represented as in (3.1) with the above L-matrix, as explained in Section 4.2 (see (4.3)). Introduce $W_{[k,l)}$ as in (3.5). By the general theory of these L-ensembles, we need to compute

(7.4) $$K = \mathbf{1} - D^{-1} + D^{-1} C M^{-1} B D^{-1}$$

for some invertible matrix M. Here, D is as in (3.9), $M = BD^{-1}C$,

(7.5) $$B = \begin{bmatrix} \Phi & 0 & 0 & \cdots & 0 \end{bmatrix},$$

(7.6) $$C = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix}.$$

The plan now is to analyse each of the different components of (7.4), namely BD^{-1}, $D^{-1}C$, $(M)^{-1}$ and the upper triangular matrix $1 - D^{-1}$.

First of all,

$$BD^{-1} = \begin{bmatrix} \Phi & \Phi W_{[0,1)} & \Phi W_{[0,2)} & \cdots & \Phi W_{[0,N)} \end{bmatrix}$$

which is an $n_0 \times |\mathcal{M}|$ matrix. We'll call the kth block of this Φ_k. That is, $\Phi_0 := \Phi$ and

(7.7) $$\Phi_k := \Phi W_{[0,k)},$$

for $k = 1, \ldots, N$, and an explicit expression for it will be given in Lemma 7.3. Next, let us look at

$$(7.8) \quad D^{-1}C = \begin{bmatrix} F_0 + W_{[0,1)}F_1 + W_{[0,2)}F_2 + W_{[0,3)}F_3 + \cdots + W_{[0,N)}F_N \\ F_1 + W_{[1,2)}F_2 + W_{[1,3)}F_3 + \cdots + W_{[1,N)}F_N \\ \vdots \\ F_N \end{bmatrix}$$

$$=: \begin{bmatrix} \bar{\Psi}_0 \\ \bar{\Psi}_1 \\ \vdots \\ \bar{\Psi}_N \end{bmatrix}$$

where, for $k = 0, \ldots, N$,

$$(7.9) \qquad \qquad \bar{\Psi}_k = F_k + \sum_{j=k+1}^{N} W_{[k,j)}F_j.$$

As mentioned it is useful to do column operations on $\bar{\Psi}_k$, which are represented by the $n_0 \times n_0$ matrix R_k, see Section 4.3. We will choose R_k in such a way that MR_k, for $k = 0, \ldots, N-1$, is asymptotically the identity matrix as $u \to -\infty$. Thus for $k = 0, \ldots, N$ let

$$(7.10) \qquad \qquad \Psi_k := \bar{\Psi}_k R_k.$$

These will be explicitly computed in Lemmas 7.4 and 7.5.

Block (k, k') of the kernel in (7.4) can with this notation be written, remembering (3.9), as

$$(7.11) \qquad [K_{k,k'}]_{x,y} = -W_{[k,k')} + \sum_{i,j=1}^{n_0} [\bar{\Psi}_k]_{x,i}[M^{-1}]_{i,j}[\Phi_{k'}]_{j,y}$$

$$(7.12) \qquad = -W_{[k,k')} + \sum_{i,j=1}^{n_0} [\Psi_k]_{x,i}[(MR_k)^{-1}]_{i,j}[\Phi_{k'}]_{j,y}.$$

Note that, as we shall see in Lemma 7.5, only the columns $1, \ldots, n_k$ of Ψ_k are non-zero. So we need only compute rows $1, \ldots, n_k$ of $(MR_k)^{-1}$. As it happens we never need to explicitly write down what R_k is but it can in principle be extracted from the proof of Lemma 7.5.

Recall from (7.4) that $M = BD^{-1}C$, but what we really need is MR_k for $k = 0, \ldots, N-1$. Multiply (7.5) with (7.8) to compute

$$MR_k = BD^{-1}CR_k = \Phi F_0 R_k + \sum_{j=1}^{N} \Phi W_{[0,j)}F_j R_k.$$

For $k = 0$ that specialises to $MR_0 = \Phi_0 \Psi_0$. For $k > 0$ it will later turn out that we only need the first n_k columns of MR_k. That allows us to remove those terms that only contribute to columns $n_k + 1$ to n_0, which by (7.3) are those that involve $F_0, \ldots, F_{k-1}$. We shall denote by $\simeq$ the operation of removing the unnecessary columns.

$$MR_k \simeq \sum_{j=k}^{N} \Phi W_{[0,j)} F_j R_k$$

$$= \Phi W_{[0,k)} (F_k + \sum_{j=k+1}^{N} W_{[k,j)} F_j) R_k$$

$$= \Phi_k \Psi_k$$

Another way of saying that is

(7.13) $$[MR_k]_{i,j} = \int_{\mathbb{R}} [\Phi_k]_{i,x} [\Psi_k]_{x,j} dx.$$

for $i, j = 1, \ldots, n_k$.

7.1. Computing $1 - D^{-1}$

From (3.9) it is clear that this is a matrix of size $N|\mathcal{M}| \times N|\mathcal{M}|$. Furthermore, the (k, l) block of this is identically 0 if $l \leqslant k$ and otherwise $-W_{[k,l)}$. Recall from (3.5) the definition of $W_{[k,l)}$. For notation introduce a function $S : \{0, \ldots, N-1\} \to \mathbb{R}$ defined by

$$S(k) := \prod_{l=1}^{n_k} r_{s_{n_0+1-l}-s_0}.$$

LEMMA 7.1.

(7.14) $$[W_{[k,k')}]_{x,z} = (r_{t_k-t_0})^{n_{k'}-n_k} \frac{S(k)}{S(k')} \int_{\mathbb{R}} H^{n_k - n_{k'}} (x-y) p_{t_{k'}-t_k}(y, z) \, dy$$

for $k < k'$ and $[W_{[k,k')}]_{x,z} = 0$ otherwise.

Recall the definition of H in (6.1). Here we will use the method, described in Section 4.1, replacing $\mathcal{M}$ by $\mathbb{R}$. Remember Section 4.1 explains how to go from the discrete to the continuous; in particular how matrix multiplications in the discrete turn to convolutions in the continuous setting.

Proof. — This is shown by induction and $k' = k + 1$ is the basic case. Let's say $k \in$ Space. (Recall from the first paragraph of Section 7 what that means.) Then $s_{n_0+1-n_k} = t_k$ and $n_k = n_{k'} + 1$ so the above expression

reduces to $[W_{[k,k+1)}]_{x,z} = H(x - z)$ which is correct. Otherwise $k \in$ Time. Then all the S and r factors in (7.14) equals one. The integral evaluates to $[W_{[k,k+1)}]_{x,z} = p_{t_{k+1}-t_k}(x,z)$ which is correct.

Of course

$$[W_{[k,k'+1)}]_{x,z} = \int_{\mathbb{R}} [W_{[k,k')}]_{x,y}[W_{k'}]_{y,z} \, dy.$$

If $k' \in$ Space then apply first (6.8) which pops out an $r_{t_{k'}-t_k} = r_{t_{k'}-t_0}/r_{t_k-t_0}$ and then apply (6.6). But $k' \in$ Space implies that $t_{k'} = s_{n_0+1-n_{k'}} = s_{n_0+1-(n_{k'+1}+1)}$. On the other hand if $k' \in$ Time, just a single application of (6.7) completes the induction.   □

Lemma 7.2.

$$(7.15)\ [W_{[k,k')}]_{x,z} = r_{t_{k'}-t_0}^{n_{k'}-n_k}\frac{S(k)}{S(k')} \sum_{l=n_{k'}+1}^{n_k} \frac{h_{n_k-l}^{(t_k)}(x)(\sigma(t_k))^{n_k-l-1}r_{t_{k'}-t_k}^{n_k-l}}{\sqrt{2(n_k-l)!}\pi^{\frac{1}{4}}}$$

$$\times \int_{\mathbb{R}} w^{(t_{k'})}(y)H^{l-n_{k'}}(y-z) \, dy$$

$$+ r_{t_{k'}-t_0}^{n_{k'}-n_k}\frac{S(k)}{S(k')}\frac{(\sigma(t_{k'}))^{n_k-n_{k'}}}{\sqrt{2}\sigma(t_k)}$$

$$\times \sum_{l=-\infty}^{n_{k'}} \sqrt{\frac{(n_{k'}-l)!}{(n_k-l)!}}q_{t_{k'}-t_k}^{(t_k),n_k-l}h_{n_k-l}^{(t_k)}(x)h_{n_{k'}-l}^{(t_{k'})}(z)w^{(t_{k'})}(z).$$

for $k' > k$ and $[W_{[k,k')}]_{x,z} = 0$ otherwise.

Proof. — Apply Lemma 6.2 and (6.11)–(6.13) to the formula in Lemma 7.1.   □

7.2. Computing BD^{-1}

Recall from (7.7) the definition of Φ_k, which we shall now compute. Multiply (7.2) with (7.15) and use the orthogonality (6.9) to get

Lemma 7.3.

$$(7.16)\quad [\Phi_{k'}]_{l,z} = r_{t_{k'}-t_0}^{n_{k'}-n_0}\frac{S(0)}{S(k')}(\sigma(t_{k'}))^{n_0-n_{k'}} \times$$

$$\times \sqrt{\frac{(n_{k'}-l)!}{(n_0-l)!}}q_{t_{k'}-t_0}^{(t_0),n_0-l}h_{n_{k'}-l}^{(t_{k'})}(z)w^{(t_{k'})}(z)$$

for $l \leqslant n_{k'}$ (equivalently $k' \geqslant \tau_{n+1-l}$) and

(7.17)

$$[\Phi_{k'}]_{l,z} = r_{t_{k'}-t_0}^{n_{k'}-n_0} \frac{S(0)}{S(k')} \frac{(\sigma(t_0))^{n_0-l} r_{t_{k'}-t_0}^{n_0-l}}{\sqrt{(n_0-l)!} \pi^{\frac{1}{4}}} \int_{\mathbb{R}} w^{(t_{k'})}(y) H^{l-n_{k'}}(y-z) \, dy$$

for $l \geqslant n_{k'}$.

7.3. Computing $D^{-1}C$

We do the same at the other end. Recall the definition of Ψ_k in (7.9). Explicit computations, similar to what we did for Φ_k in the preceding section, leads us to conjecture a general expression, which is proved by induction.

LEMMA 7.4.

$$(7.18) \quad [\bar{\Psi}_k]_{x,l} = \left(\prod_{j=l}^{n_k} r_{s_{n_0+1-j}-t_k} \right) \int_{\mathbb{R}} H^{n_k+1-l}(x-y) p_{s_{n_0+1-l}-t_k}(y,u) \, dy$$

for $n_k \geqslant l$ (equivalently $k \leqslant \tau_{n_0+1-l}$) and $[\bar{\Psi}_k]_{x,l} \equiv 0$ otherwise.

Proof. — To get the hang of it, consider column one of $\bar{\Psi}_k$ for all $k = 0, 1, \ldots$. Recall (7.3). Of course F_N is quite tame, it is identically zero giving us $\bar{\Psi}_N = 0$. By definition $\tau_{n_0} = N - 1$, so step number $N - 1 \in$ Space and $\bar{\Psi}_{N-1} = F_{N-1}$ is non-zero. Remember that F_{N-1} is non-zero only in column one and it is the only one of $F_0, \ldots, F_{N-1}$ that is non-zero in that column. So

$$[\bar{\Psi}_k]_{x,1} = [W_{[k,N-1)} F_{N-1}]_{x,1}.$$

Repeated applications of (6.8) and (6.6) specialises this to (7.18) if you keep track of the constants.

More generally, for $k = 0, \ldots, N$, recall from (7.3) that column l of F_k is only non-zero if $k = \tau_{n_0+1-l}$. Remember too from (7.9) that F_k only occurs in $\bar{\Psi}_0, \ldots, \bar{\Psi}_k$. That accounts for the fact that $[\bar{\Psi}_k]_{x,l} \equiv 0$ for $k > \tau_{n_0+1-l}$ (equivalently $n_k < l$).

The induction on k is now done backwards. Fix $l \in \{1, \ldots, n_0\}$. We shall show that the formula (7.18) is true for this particular value of l. Specialising to $\bar{k} = \tau_{n_0+1-l}$, equivalently $t_{\bar{k}} = s_{n_0+1-l}$, gives us $[\bar{\Psi}_{\bar{k}}]_{x,l} = r_0 H^1(x-u) = F_{\bar{k}}$. Assume now that the theorem gives the correct expression for $\bar{\Psi}_{k+1}$. Since $k \neq \bar{k} + 1$ it is clear that $F_{k+1} = 0$ in column l. Also, $W_k W_{[k+1,\bar{k})} = W_{[k,\bar{k})}$ So

$$(7.19) \qquad [\bar{\Psi}_k]_{x,l} = \int_{\mathbb{R}} [W_k]_{x,y} [\bar{\Psi}_{k+1}]_{y,l} \, dy.$$

If $k \in$ Time then applying first (6.8) and then (6.12) completes the proof. If $k \in$ Space then (6.6) does the same. The calculation is in reality only a question of keeping track of the coefficients. □

We now exercise the right to do column operations. Recall from (7.10) the definition of Ψ_k. R_k is chosen judiciously to give a very simple expression for MR_k.

LEMMA 7.5. — There exists an R_k such that for $x \in \mathcal{M}$ and u a large negative real number,

$$[\Psi_k]_{x,l} = h_{n_k-l}^{(t_k)}(x) r_{t_k-t_0}^{n_0-n_k} q_{t_k-t_0}^{(t_0),l-n_0} \times$$

$$\times \sqrt{\frac{(n_0-l)!}{(n_k-l)!} \frac{S(k)}{S(0)} \frac{(\sigma(t_k))^{n_k-n_0}}{\sqrt{2}\sigma(t_k)}} + O(e^{-u^2/2})$$

for $l = 1, \ldots, n_k$, and $[\Psi_k]_{x,l} = 0$ for $l = n_k+1, \ldots, n_0$.

Proof. — This bit is rather tricky. Fix k and let $\rho_l = s_{n_0+1-n_k+l} - t_k$. Let

$$f_l(x, u) = C(l)[\bar{\Psi}_k]_{x,n_k-l}$$

for $l \in \{0, \ldots, n_k-1\}$ where $C(l)$ is the constant that gives

$$f_l(x, u) = \int_{\mathbb{R}} H^{l+1}(x-y) p_{\rho_l}(y, u)\, dy$$

$$= \int_{\mathbb{R}} H^{l+1}(x-y) p_{\rho_l}(y - r_{\rho_l}^{-1}u, 0)\, dy$$

$$= \int_{\mathbb{R}} H^{l+1}(x-y - r_{\rho_l}^{-1}u) p_{\rho_l}(y, 0)\, dy.$$

The second equality is due to (6.10) and the third one is a change of variables $y \mapsto y + r_{\rho_l}^{-1}u$. Now what we need to do is to produce a sequence of functions $g_0, \ldots, g_{n_k-1}$ such that for each l,

(1) g_l is a linear combination of $f_0, \ldots, f_l$,
(2) $g_l(x, u)$ differs from a polynomial in x of order l by at most a term exponentially decreasing in u. That is, there is a polynomial $\bar{h}_l(x)$ of degree l such that

$$g_l(x, u) = \bar{h}_l(x) + O(e^{-u^2/2}).$$

Just to see how it works consider

$$f_0(x, u) = \int_{\mathbb{R}} H(x - y - r_{\rho_0}^{-1}u) p_{\rho_0}(y, 0)\, dy$$

$$= \int p_{\rho_0}(y, 0)\, dy + O(e^{-u^2/2})$$

as $u \to -\infty$. So let $g_0 = f_0$ implying that h_0 is a constant.

Now for some constant

$$C = \left(\int_{\mathbb{R}} p_{\rho_1}(y,0)\, dy \right) \left(\int_{\mathbb{R}} p_{\rho_0}(y,0)\, dy \right)^{-1}$$

look at

$$f_1(x,u) - Cr_{\rho_1}^{-1} u f_0(x,u) = \int (x - y - r_{\rho_1}^{-1} u) H(x - y - r_{\rho_1}^{-1} u) p_{\rho_1}(y,0)$$

$$- Cr_{\rho_1}^{-1} u \int H(x - y - r_{\rho_1}^{-1} u) p_{\rho_0}(y,0)$$

(where C is chosen so that, in the limit of large negative u, the integrands partially cancel out.)

$$= \int (x - y) p_{\rho_1}(y,0)\, dy + O(e^{-u^2/2})$$

$$= x \int p_{\rho_1}(y,0)\, dy - \int y p_{\rho_1}(y,0)\, dy + O(e^{-u^2/2}).$$

Thus set $g_1(x,u) = f_1 - Cr_{\rho_1}^{-1} u f_0$ and $\bar{h}_1$ accordingly.

Now that we see it works for $l = 0$ and 1, let's do the induction. Say the statement is true for l, then

$$f_{l+1}(x,u) = \int (x - y - r_{\rho_{l+1}}^{-1} u)^{l+1} H(x - y - r_{\rho_{l+1}}^{-1} u) p_{\rho_{l+1}}(y,0)\, dy$$

$$= \sum_{i=0}^{l+1} \binom{l+1}{i} \int (x-y)^{l+1-i} (r_{\rho_{l+1}}^{-1} u)^i H(x-y-r_{\rho_{l+1}}^{-1}) p_{\rho_{l+1}}(y,0)\, dy.$$

The first term is already a polynomial in x of degree $l+1$ modulo an O-term. The other terms are polynomials in x of lower degree which by the induction hypothesis can be expressed in $g_0, \ldots, g_l$.

What we have effectively done is that we have taken various linear combinations, encoded in R, of columns of $\bar{\Psi}_k$ producing

$$[\bar{\Psi}_k R]_{x,l} = \text{polynomial of degree } l + \text{perturbation exponentially small in } u.$$

Now we will go ahead and do more column operations and multiply the columns with suitable constants. Let R' be the matrix that encodes the column operations and multiplications that turn the columns of $\bar{\Psi}_k R$ into Hermite polynomials with factors as in the statement of Lemma 7.5. Set $R_k = RR'$ and $\Psi_k = \bar{\Psi}_k R_k$ and we are done. $\qquad \square$

7.4. Computing $(MR_k)^{-1}$

First observe that M is essentially lower triangular. To see this, for $1 \leqslant i < j \leqslant n_0$ consider

$$(7.20) \qquad [M]_{ij} = \int_{\mathbb{R}} [\Phi_0]_{i,x} [\bar{\Psi}_0]_{x,j} \, dx = O(e^{-u^2/2}).$$

This is a computation. You insert the expressions for Φ_0 and Ψ_0, apply first (6.6), then then (6.4) and lastly perform $n_0 - j$ applications of (6.5). By (6.9) the conclusion in (7.20) holds.

Let I_n be the identity matrix of size $n \times n$. In block form M and R_k are:

$$MR_k = \begin{bmatrix} * & O(e^{-u^2/2}) \\ * & * \end{bmatrix} \begin{bmatrix} * & 0 \\ 0 & I_{n_0-n_k} \end{bmatrix} = \begin{bmatrix} I_{n_k} + O(e^{-u^2/2}) & O(e^{-u^2/2}) \\ * & * \end{bmatrix}.$$

The top-right block of MR_k is essentially I_{n_k} by construction. To see that, recall (7.13), insert the results in Lemmas 7.3 and 7.5 and apply (6.9). To invert this matrix, use (3.8) with $M = -I_{n_k} + O(e^{-u^2/2})$ and $B = O(e^{-u^2/2})$ and we are only interested in the top two blocks.

PROPOSITION 7.6.

$$[(MR_k)^{-1}]_{ij} = \delta_{i,j} + O(e^{-u^2/2})$$

for $i = 1, \ldots, n_k$ and $j = 1, \ldots, n_0$.

Essentially, it is diagonal due to our judicious choice of polynomials and furthermore the identity matrix due to our choice of factors in Lemma 7.5.

7.5. Putting it all together

Recall the expression for the kernel in (7.12). For the case $k \geqslant k'$, inserting the results of Lemmas 7.3 and 7.5, (6.11)–(6.13), taking account of the cancellations and letting $u \to -\infty$ gives

$$(7.21) \quad K((n_k, x, t_k), (n_{k'}, x', t_{k'})) = r_{t_{k'}-t_0}^{n_{k'}-n_k} \frac{S(k)}{S(k')} \frac{(\sigma(t_{k'}))^{n_k-n_{k'}}}{\sqrt{2}\sigma(t_k)} \times$$

$$\times \sum_{l=1}^{n_k} \sqrt{\frac{(n_{k'}-l)!}{(n_k-l)!}} q_{t_{k'}-t_k}^{(t_k),n_k-l} h_{n_k-l}^{(t_k)}(x) h_{n_{k'}-l}^{(t_{k'})}(x') w^{(t_{k'})}(x').$$

In this case you don't need to use (7.17).

In the case $k < k'$ we also need Lemma 7.2. The first n_k terms in the summand in (7.12) cancel the corresponding terms in (7.15) and we are left with the following infinite sum.

$$(7.22) \quad K((n_k, x, t_k), (n_{k'}, x', t_{k'})) = -r_{t_{k'}-t_0}^{n_{k'}-n_k} \frac{S(k)}{S(k')} \frac{(\sigma(t_{k'}))^{n_k-n_{k'}}}{\sqrt{2}\sigma(t_k)} \times$$

$$\times \sum_{l=-\infty}^{0} \sqrt{\frac{(n_{k'}-l)!}{(n_k-l)!}} q_{t_{k'}-t_k}^{(t_k),n_k-l} h_{n_k-l}^{(t_k)}(x) h_{n_{k'}-l}^{(t_{k'})}(x') w^{(t_{k'})}(x').$$

We shall now specialise to the case of Dyson Brownian motion and Warren respectively.

Proof of Theorem 1.1. — Insert the various expressions in section 6.1 into (7.21) and (7.22) and multiply with the conjugating factors

$$\frac{S(k')e^{n'(t'-t_0)}}{S(k)e^{n(t-t_0)}}$$

which always cancel out when you take determinants.

To show that the integral representation equals this shifted sum of Hermite polynomials, observe that

$$\frac{1}{e^{-(t'-t)}v - u} \frac{v^{n'}}{u^n} = \frac{v^{n'}}{u^{n+1}} \sum_{l=0}^{\infty} \left(\frac{u}{e^{-(t'-t)}v}\right)^l$$

and plug in the integral representations in Section 6.3. The function ϕ should be expanded according to Lemma 6.2. □

Proof of Theorem 1.2. — Insert the various expressions in section 6.2 into (7.21) and (7.22) and multiply with

$$(7.23) \qquad \frac{S(k')}{S(k)} 2^{(n'-n)/2}$$

which always cancels out when you take determinants. The integral representation is done exactly as in the last proof. □

8. Asymptotics

We shall now study the scaling limit of the Dyson Brownian Minor kernel from (1.5) in the bulk.

Proof of Theorem 1.3. — This is done by saddle point analysis. Let $\tau_N = e^{-(t'-t)/2N}$. We start out with the case $(n, t) \geqslant (n', t')$. Use the

representation (1.5) and substitute $u \mapsto u\sqrt{N/2}$ and $v \mapsto \tau_N^{-1}v\sqrt{N/2}$ in the integral. We must now compute

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_\gamma du \int_\Gamma dv \frac{\tau_N^{-N-n'-1}}{v-u} \frac{v^{n'}}{u^n} \exp\Big[(N/2)\big(\tau_N^{-2}v^2 - u^2$$

$$+ 4a(u - \tau_N^{-1}v) + 2\ln v - 2\ln u\big) + ux - \tau_N^{-1}vy\Big].$$

The part of the exponent that is multiplied by $(N/2)$ is $f(u) - f(v)$ for

$$f(u) = -u^2 + 4au - 2\ln u.$$

The saddle points, satisfying the equation $f'(u) = 0$ are given by (1.9) and are both on the unit circle. We let θ be the angle specifying those points, so that $u_\pm = e^{\pm i\theta}$. The Taylor expansions around these critical points are

$$f(u_\pm + h) = f(u_\pm) + (\mp 4a\sqrt{1-a^2} + 4(1-a^2)i)h^2 + o(h^2).$$

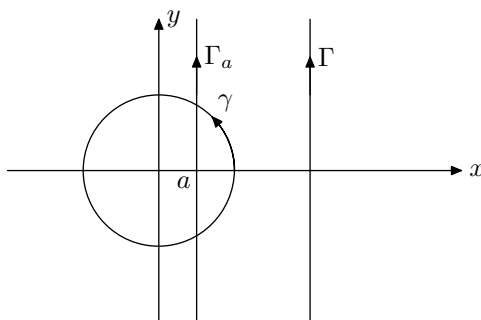The only thing we really need is that the coefficient of h^2 is non-zero.

Figure 8.1. New contours of integration.

Now deform the contour Γ to Γ_a which is a straight line from $a - i\infty$ to $a + i\infty$. See Figure 8.1. When deforming the v-contour through the u-contour out pops the residue at $u = v$ which is exactly

$$\frac{1}{2\pi i} \int_{u_-}^{u_+} u^{n'-n} e^{\frac{1}{2}(t'-t)(u^2-2au)+u(x-y)} \, du.$$

It remains to show that all the other parts of the contours evaluate to zero.

To see what happens around the saddle point u_+ perform the change of variables $u \mapsto u_+ + u/\sqrt{N}$ and $v \mapsto u_+ + v/\sqrt{N}$.

$$\frac{1}{2\pi i} \int du \int dv \frac{\tau_N^{-N-n'-1}}{v-u} \frac{(u_+ + v/\sqrt{N})^{n'}}{(u_+ + u/\sqrt{N})^n} \times$$

$$\times \exp\left[\frac{1}{4} f''(u_+)(u^2 - v^2) + u_+(x - \tau^{-1}y) + O(1 - \tau_N) + O(N^{-\frac{1}{2}})\right]$$

which, as $N \to \infty$, tends to

$$\frac{1}{2\pi i} \int du \int dv \frac{e^{\frac{1}{2}(t'-t)}}{v-u} (u_+)^{n'-n} \exp\left[\frac{1}{4} f''(u_+)(u^2 - v^2) + u_+(x - y)\right].$$

Switching $u \mapsto -u$ and $v \mapsto -v$ makes the integrand change sign. Thus the integral must be zero. The saddle point at u_- contributes zero by the same argument.

For the remaining contours we need to show that $g(u, v) := \operatorname{Re} f(u) - f(v) < 1$. Well, if $g(u, v) > 1$ somewhere then the integral would not be convergent. But the whole expression is a probability density and must therefore be finite, so this is a contradiction. If at some point (u, v) it happens that $g(u, v) = 1$ then $f'(u) \neq 0$, for we have already accounted for all saddle points. Thus by the definition of derivative there must be a point nearby where $g(u, v) > 1$ which is a contradiction.

Next we need to compute the same scaling limit of the function ϕ from (1.6) which is a very straight forward computation, yielding (1.8). $\quad\square$

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