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QUADRO-QUADRIC CREMONA TRANSFORMATIONS IN LOW DIMENSIONS VIA THE JC-CORRESPONDENCE

by Luc PIRIO & Francesco RUSSO (*)

Abstract. — It has been previously established that a Cremona transformation of bidegree (2,2) is linearly equivalent to the projectivization of the inverse map of a rank 3 Jordan algebra. We call this result the "*JC-correspondence*". In this article, we apply it to the study of quadro-quadric Cremona transformations in low-dimensional projective spaces. In particular we describe new very simple families of such birational maps and obtain complete and explicit classifications in dimension 4 and 5.

Résumé. — Il a été établit précédemment qu'une transformation de Crémona de bidegré (2,2) est linéairement équivalente à la projectivation de l'inversion d'une algèbre de Jordan de rang 3. Ce résultat (appelé la "*correspondance JC*") est utilisé dans le présent article pour étudier les transformations birationnelles quadro-quadriques des espaces projectifs de petite dimension. En particulier, nous décrivons de nouvelles familles très simples de telles applications birationnelles et nous obtenons leur classifications complètes et explicites en dimension 4 et 5.

Introduction

The study of Cremona transformations is a quite venerable subject which received a lot of classical contributions (we refer to [37] for a rich overview of the field up to 1928, to the classical reference [17] and to [36] for an approach to the subject with classical methods but with a view to higher dimensional cases). More recently there has been a renewed interest in the subject, also for the relations with complex dynamics which inspired the solution to the longstanding problem about the simplicity of the group

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of Cremona transformations in the plane, answered with the negative in [6]. The situation in the plane can be considered quite clear, also considering the classification of finite subgroups of the Cremona group in the plane completed recently by Blanc (see [3] and the references therein). In higher dimension the structure is more complicated and less clear. For instance, the generators of the Cremona groups are unknown in contrast to the case of the plane where the classical Noether-Castelnuovo Theorem assures that the ordinary quadratic transformation and projective transformations generate the whole group. Thus since the very beginning the study of Cremona transformations in higher dimensional projective spaces was developed only for particular classes (see for example [10, 27, 28]) or for specific degrees of the homogeneous polynomials defining the map. From this point of view the first cases of interest are those defined by quadratic polynomials (see [35, 5, 28]) and thus the *simplest* examples of Cremona transformations, different from projective automorphisms, are those whose inverse is also defined by quadratic polynomials, dubbed *quadro-quadric* Cremona transformations. In [28] one finds the complete classification of quadratic Cremona transformations of $\mathbb{P}^3$ while [35] considers the case of $\mathbb{P}^4$, describing the general base locus scheme of these Cremona maps (see also the discussion in Section 3.4.1). The more recent paper [5] deals with the classification of general quadro-quadric Cremona transformations in $\mathbb{P}^4$ and $\mathbb{P}^5$ and provides some series of examples in arbitrary dimension.

A completely new approach to the subject of quadro-quadric Cremona transformations was begun in [32] and completed in [31], where general results and structure Theorems for these maps were presented. Indeed, in [31], we proved that for every $n \geqslant 3$, there are equivalences between:

- irreducible n-dimensional non degenerate projective varieties $\boldsymbol{X} \subset \mathbb{P}^{2n+1}$ different from rational normal scrolls and 3-covered by rational cubic curves, up to projective equivalence;
- n-dimensional complex **J**ordan algebras of rank three, up to isomorphisms;
- quadro-quadric **C**remona transformations of $\mathbb{P}^{n-1}$, up to linear equivalence[(1)].

[(1)] Two Cremona maps f_1, f_2 of $\mathbb{P}^n$ are said to be *linearly equivalent* if $f_2 = g \circ f_1 \circ h$ for some linear automorphisms $g, h \in \mathrm{Aut}(\mathbb{P}^n)$.

The equivalence between these sets has been named the '*XJC-correspondence*' in [31][(2)]. In this text, we use mainly one part of this correspondence, which we call the '*JC-correspondence*'. Essentially, it asserts that for every $n \geqslant 3$, *any quadro-quadric Cremona transformation* $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ *is linearly equivalent to the projectivization of the adjoint map* $x \mapsto x^{\#}$ *of a* n*-dimensional rank 3 Jordan algebra, the isomorphism class of which is uniquely determined by the linear equivalence class of* f (see Section 1 for notation and definitions).

The JC-correspondence does not only offer a conceptual interest but also provides an effective way to study quadro-quadric Cremona transformations. Indeed, the theory of Jordan algebras is now well developed and formalized, so several quite deep results and powerful algebraic tools to study them conceptually and/or effectively are now available.

Armed with the powerful algebraic machinery of the theory of Jordan algebras, in this paper we

- describe a general algebraic method to construct new quadro-quadric Cremona transformations starting from known ones (Section 2.1);
- use the above method to construct, starting from the standard Cremona transformation of $\mathbb{P}^2$, a very simple but new countable family of quadro-quadric Cremona transformations of $\mathbb{P}^n$ for arbitrary $n \geqslant 2$ (Section 2.2.1);
- use the previous construction to produce continuous families of quadro-quadric Cremona transformations starting from those associated to simple rank three Jordan algebras (Section 2.2.3);
- describe two distinct general constructions of cubo-cubic Cremona transformations starting from a quadro-quadric one (Section 2.3);
- give complete classifications of rank 3 Jordan algebras in dimension 3, 4 and 5 (in Sections 3.2, 3.3 and 3.4) and deduce from them the complete classifications, up to linear equivalence, of quadro-quadric Cremona transformations of $\mathbb{P}^n$ for $n = 2, 3$ and 4 (see TABLE 2, TABLE 3 and TABLE 6 respectively);

(2) Note that Jordan algebras are considered up to isomorphisms in the present paper whereas they were originally considered up to isotopies in [31]. If the later setting is more natural from a categorical point of view (see [31, Remark 4.2]), the XJC-correspondence still holds true at the set-theoretical level when considering rank 3 Jordan algebras up to isomorphisms. The reason behind this is the fact that over the field of complex numbers, two isotopic Jordan algebras are actually isomorphic (cf. [25, Problem 7.2.(6)]).

- provide a detailed analysis of the quadro-quadric Cremona transformations of $\mathbb{P}^4$ (in Section 3.4). For each Cremona transformation of TABLE 6, we describe as geometrically and concisely as possible the associated base locus scheme and the associated homaloidal system of quadric hypersurfaces, determine its type and its multidegree (see TABLE 7, TABLE 8 and TABLE 9). We also offer pictures of these base locus schemes (see FIGURE 5, FIGURE 6);
- give (without proof) a complete list of involutorial normal forms for elements of quadro-quadric Cremona transformations of $\mathbb{P}^5$ (cf. TABLE 10).

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1. Notation and definitions

1.0.1. General notation and definitions

If $f = [f_0 : \cdots : f_n] : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a Cremona transformation, then $\mathcal{B}_f \subset \mathbb{P}^n$ will be the *base locus scheme of* f, that is the scheme defined by the $n+1$ homogeneous forms $f_0, \ldots, f_n$ that are assumed without common factor. If all these are of degree $d_1 \geqslant 1$ and if the inverse $f^{-1} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of f is defined by forms of degree d_2 without common factor, we say that f has *bidegree* (d_1, d_2) and we put $\mathrm{bdeg}(f) = (d_1, d_2)$. The set of Cremona transformations of $\mathbb{P}^n$ of bidegree (d_1, d_2) will be indicated by $\mathbf{Bir}_{d_1,d_2}(\mathbb{P}^n)$.

By definition, a *quadro-quadric Cremona transformation* is just a Cremona transformation of bidegree $(2, 2)$.

Two Cremona transformations $f, g \in \mathbf{Bir}_{d_1,d_2}(\mathbb{P}^n)$ are said *linearly equivalent* (or just *equivalent* for short) if there exist projective transformations $\ell_1, \ell_2 : \mathbb{P}^n \to \mathbb{P}^n$ such that $g = \ell_1 \circ f \circ \ell_2$.

We define the *type of a Cremona transformation* $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ as the irreducible component of the Hilbert scheme of $\mathbb{P}^n$ to which $\mathcal{B}_f$ belongs. We put $P_i(t) = \binom{t+i}{i}$ so that $P_0 = 1$, $P_1(t) = t + 1$ and $P_2(t) = \frac{(t+2)(t+1)}{2}$.

By definition, a *Jordan algebra* is a commutative complex algebra J with a unity e such that the *Jordan identity* $x^2(xy) = x(x^2y)$ holds for every $x, y \in J$ (see [4, 18, 25]). Here we shall assume too that J is finite dimensional. It is well known that a Jordan algebra is power-associative. By

definition, the *rank* $\mathrm{rk}(J)$ of J, also called the *degree* of J in the classical literature (such as [18]), is the complex dimension of the (associative) subalgebra $\langle x\rangle$ of J spanned by the unity e and by a general element $x \in J$.

The simplest examples of Jordan algebras are those constructed from associative algebras. Let A be a non-necessarily commutative associative algebra with a unity. Denote by A^+ the vector space A with the symmetrized product $a \cdot a' = \frac{1}{2}(aa' + a'a)$. Then A^+ is a Jordan algebra. Note that $A^+ = A$ if A is commutative.

By $\mathcal{J}^m_{Q,r}$ we will indicate a m-dimensional direct sum $\mathbb{C} \oplus W$ endowed with the Jordan product $(\lambda, w) \cdot (\lambda', w') = (\lambda\lambda' - Q(w, w'), \lambda w' + \lambda' w)$, where $Q(\cdot, \cdot)$ stands for the polarization of a quadratic form Q on W, of rank r. When we write $\mathcal{J}^m_Q$ we will assume that $W = \mathbb{C}^{m-1}$ and that $Q(x) = \sum_{i=1}^{m-1} x_i^2$ in the standard system of coordinates $x = (x_1, \ldots, x_{m-1})$ on $\mathbb{C}^{m-1}$. Otherwise when the integer $r < m-1$ is specified we shall assume that $Q(x) = \sum_{i=1}^{r} x_i^2$. Except when $W = 0$, a Jordan algebra $\mathcal{J}^m_{Q,r}$ has rank 2.

A Jordan algebra of rank 1 is isomorphic to $\mathbb{C}$ (with the standard multiplicative product). It is classical and easy to prove that any rank 2 Jordan algebra is isomorphic to an algebra $\mathcal{J}^m_{Q,r}$ defined above (cf. [4, Satze 7.1]). In this paper, we will mainly consider Jordan algebras of rank 3. These are the simplest Jordan algebras which have not been yet classified in arbitrary dimension. Due to the JC-correspondence their classification is equivalent to that of quadro-quadric Cremona transformations in arbitrary dimension, which shows the complexity of the problem.

Let J be a rank 3 Jordan algebra. The general theory (as presented on page 221 of [18]) specializes in this case (cf. [25, II. 4]) and ensures the existence of a linear form $T : J \to \mathbb{C}$ (the *generic trace*), of a quadratic form $S \in \mathrm{Sym}^2(J^*)$ and of a cubic form $N \in \mathrm{Sym}^3(J^*)$ (the *generic norm*) such that $x^3 - T(x)x^2 + S(x)x - N(x)e = 0$ for every $x \in J$. Moreover, x is invertible in J if and only if $N(x) \neq 0$ and in this case $x^{-1} = N(x)^{-1}x^{\#}$, where $x^{\#}$ stands for the *adjoint* of x defined by $x^{\#} = x^2 - T(x)x + S(x)e$. The adjoint satisfies the identity $(x^{\#})^{\#} = N(x)x$.

The algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries is associative hence $M_n(\mathbb{C})^+$ is a Jordan algebra. According to Cayley-Hamilton Theorem, it is of rank n, the generic trace of $M \in M_n(\mathbb{C})^+$ is the usual one, $N(M) = \det(M)$ and the adjoint is the matricial one, that is the transpose of the cofactor matrix of M.

For $x = (\lambda, w) \in \mathcal{J}^m_{Q,r}$, one has $x^2 - T(x)x + N(x)e = 0$ with $T(x) = 3\lambda$ and $N(x) = \lambda^2 + q(w)$. Thus it can be verified that $x^{\#} = (\lambda, -w)$ in this case.

More generally let J be a power-associative algebra with $r = \mathrm{rk}(J) \geqslant 2$. Then defining similarly the adjoint of an element $x \in J$, the identity $\left(x^{\#}\right)^{\#} = N(x)^{r-2}x$ holds so that the projectivization of the adjoint $[\#] : \mathbb{P}(J) \dashrightarrow \mathbb{P}(J)$ is a birational involution of bidegree $(r-1, r-1)$ on $\mathbb{P}(J)$.

The inverse map $x \mapsto x^{-1} = N(x)^{-1}x^{\#}$ on J naturally induces a birational involution $\widetilde{\jmath} : \mathbb{P}(J \times \mathbb{C}) \dashrightarrow \mathbb{P}(J \times \mathbb{C})$ of bidegree (r, r), defined by $\widetilde{\jmath}([x, r]) = [rx^{\#}, N(x)]$. Such maps were classically investigated by N. Spampinato and C. Carbonaro Marletta, see [38, 22, 23], who produced examples of interesting Cremona involutions in higher dimensional projective spaces. It is easy to see that setting $\widetilde{J} = J \times \mathbb{C}$, then for $(x, r) \in \widetilde{J}$ one has $(x, r)^{\#} = (rx^{\#}, N(x))$ so that the map $\widetilde{\jmath}$ is the adjoint map of the algebra $\widetilde{J}$. A Cremona transformation of bidegree (r, r) will be called *of Spampinato type* if it is linearly equivalent to the adjoint of a direct product $J \times \mathbb{C}$ where J is a power-associative algebra of rank $r \geqslant 2$. In [8] and [9], Del Pezzo studied examples of quadro-quadric Cremona transformations in $\mathbb{P}^3$ and in $\mathbb{P}^4$. From our point of view, these examples are part of the series of Spampinato type Cremona transformations on $\mathbb{P}^n$ associated to $\mathcal{J}^n_{Q,0}$, i.e. they are linearly equivalent to the projectivizations of the involutions of the rank three Jordan algebras $\mathbb{C} \times \mathcal{J}^n_{Q,0}$.

We will denote by $J, J', \mathcal{J}, \ldots$ complex Jordan algebras of finite dimension and $J \times J'$ will stand for the direct product of Jordan algebras. By definition, the *radical* $\mathrm{Rad}(J)$ of J is its maximal solvable ideal (cf. [18, p. 192]). It will be denoted by R when no confusion could arise . The *semi-simple part of* J, denoted by J_{ss}, is the quotient J/R of J by its radical. As its name suggests, it has no radical, *i.e.* $\mathrm{Rad}(J_{\mathrm{ss}}) = 0$.

By $\mathcal{B}(J) \subset \mathbb{P}(J)$ we shall indicate the base locus scheme of the birational involution $[\#]$ associated to J and $\mathcal{I}_{\mathcal{B}(J)}$ will be the graded ideal generated by the homogenous polynomials defining the adjoint of J.

1.0.2. Some non-reduced punctual schemes in projective spaces

For integers n, k such that $0 \leqslant k < n$ and for any nonnegative integer t, one defines a *t-multiple $\mathbb{P}^k$ in a $\mathbb{P}^n$* as a subscheme of $\mathbb{P}^n$ with homogeneous ideal equal to the t-th power of the ideal of a linear subspace of dimension k in $\mathbb{P}^n$.[(3)]

[(3)] This terminology is not standard. For instance, it does not correspond exactly to the one used in [11].

We shall also provide some pictures of the base locus schemes of quadro-quadric Cremona maps in dimension $2, 3$ and 4. Many of these base loci contain a non-reduced punctual scheme as an irreducible component of particular type, which we now describe and picture in this subsection.

– *Scheme* τ. If $\tau_0 = \mathrm{Proj}\big(\mathbb{C}[a,b]/(b,a^2)\big)$ is the length 2 affine scheme $\mathrm{Spec}\big(\mathbb{C}[\epsilon]/(\epsilon^2)\big)$ naturally embedded in $\mathbb{C} \subset \mathbb{P}^1$, we will designate by τ any scheme in a projective space $\mathbb{P}^n$ obtained as the image $\iota_*(\tau_0)$ for a linear embedding $\iota : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. More intuitively, τ is a punctual scheme in a projective space formed by a point p plus another point infinitely near to it. Since geometrically τ is nothing but a tangent direction at one point, this scheme will be pictured as follows in the sequel:

Figure 1. Pictural representation of τ

– *Scheme* ξ. Let $\xi_0 \simeq \mathrm{Spec}(\mathbb{C}[a,b]/(a^2,ab,b^2))$ be a double point in $\mathbb{P}^2$. We will designate by ξ any scheme in a projective space $\mathbb{P}^n$ obtained as the image $\iota_*(\xi_0)$ for a linear embedding $\iota : \mathbb{P}^2 \hookrightarrow \mathbb{P}^n$. Intuitively, τ is a punctual scheme in a projective space formed by a point p plus two distinct points infinitely near of p. Geometrically τ is nothing but two tangent directions of a projective space at a given point, it is reasonable to picture this scheme as follows in the sequel:

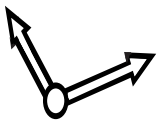


Figure 2. Pictural representation of ξ

The reader has to be aware that this representation is a bit misleading: indeed, if p stands for the support of ξ, there is no distinguishable tangent direction at p associated to ξ. What is intrinsically attached to ξ is the pencil of tangent directions spanned by the two represented in Figure 2. The projective tangent space of ξ at p coincides with the 2-plane $\langle \xi \rangle$ spanned by ξ.

– *Scheme* η. Let $\eta_0 = \mathrm{Proj}\big(\mathbb{C}[a,b]/(b,a^3)\big)$ be the length 3 affine scheme $\mathrm{Spec}\big(\mathbb{C}[\epsilon]/(\epsilon^3)\big)$ naturally embedded in $\mathbb{C} \subset \mathbb{P}^1$. We will designate by η any scheme in a projective space $\mathbb{P}^n$ obtained as the image $\nu_*(\eta_0)$ for a quadratic

Veronese embedding $\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. More intuitively, η is a punctual scheme in a projective space formed by a point $p = \eta_{\mathrm{red}}$ lying on a conic $C \subset \mathbb{P}^n$, plus another point p' infinitely near to p on C, plus a third point p'' still on C, infinitely near to p'. In geometrical terms η is nothing but an osculating flag of the second order[(4)] to a smooth conic, thus this scheme will be pictured as follows in the sequel:

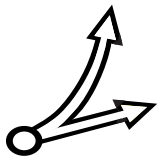


Figure 3. Pictural representation of η

By definition, the *tangent line to* η, noted by $T_p\eta$, is the line determined by p and p', which is included in the 2-plane $\langle\eta\rangle \subset \mathbb{P}^n$ spanned by η, which is also the projective tangent space to η at p.

Let H be a hypersurface of $\mathbb{P}^n$. We will say that H *osculates* (or *is osculating*) along η if $\eta \subset H$ as subschemes of $\mathbb{P}^n$. More geometrically, this means that if $C' \subset \mathbb{P}^n$ is any curve passing through p with second osculating flag at this point equal to the one associated to η, then the intersection multiplicity between C' and H at p is at least 3.

– *Scheme* χ. Let's define

$$\chi = \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]/(x^2, xy, y^2, z^2, xz, 2yz - xt)\big) \subset \mathbb{P}^3.$$

This scheme can be characterized from an algebraic and geometric point of view as follows: according to [26, p. 445], up to projective equivalence, there exist only two punctual degree four schemes in $\mathbb{P}^3$ defined by six quadratic equations. The projective tangent space at the supporting point of one of them is the whole $\mathbb{P}^3$ while χ is the other one. By definition, the *tangent plane to* χ is the projective tangent space to χ at the supporting point $p = \chi_{\mathrm{red}}$. It is a 2-plane denoted by $T_p\chi$ that is strictly contained in the linear span $\langle\chi\rangle = \mathbb{P}^3$ of χ.

More intuitively and geometrically, χ can be obtained as follows: let ℓ be a line in $\mathbb{P}^3$ that is transverse to the scheme η described just above (this means that ℓ passes through $p = \eta_{\mathrm{red}}$ and is distinct from $T_p\eta$). For every

(4) The *osculating flag of order* k of a (reduced) curve $C \subset \mathbb{P}^n$ at a smooth point c is the flag $\{c\} = T_c^{(0)}C \subset T_c^{(1)}C \subset \cdots \subset T_c^{(k)}C$ where for any non-negative integer l, $T_c^{(l)}C \subset \mathbb{P}^n$ stands for the osculating space of order l to C at c.

$x \in \ell \setminus \{p\}$, let χ_x be the union of η with x. Then clearly, χ is the flat limit of the schemes χ_x when x tends to p.

This geometrical description of χ explains that this subscheme will be pictured as follows in the sequel:

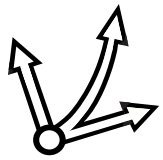


Figure 4. Pictural representation of χ

The reader has to be aware that this picture is a bit misleading: the presence of two rectilinear arrows in it does not mean that there are two distinguishable tangent directions at p associated to χ. What is intrinsically attached to χ is the 2-plane spanned by the represented two tangent directions, which is nothing but the tangent plane to χ.

Let $H \subset \mathbb{P}^3$ be a surface. The schematic condition that H contains a scheme χ translates geometrically as the fact that H is tangent to a 2-plane π at a fixed point $p \in \pi$ and osculates at order two to a smooth conic C tangent to π at this point.

2. Constructions and new families of quadro-quadric Cremona transformations

We now use some theory of non-associative algebras to construct algebraically or/and geometrically new families of involutorial Cremona transformations in arbitrary dimensions.

2.1. Construction of new quadro-quadric Cremona transformations from known ones

As far as we know, very little is known about explicit examples of quadro-quadric Cremona transformations of $\mathbb{P}^m$, at least for large enough m. In fact, there is almost no description in the literature of such birational maps in higher dimension. A notable exception is the series of quadratic elementary Cremona transformations (and their possible degenerations), that have been considered in arbitrary dimension by classical authors. They are

obtained easily from an irreducible quadric hypersurface in $\mathbb{P}^{m+1}$ by two different projections from smooth points. Recently, Bruno and Verra have described (*cf.* Proposition 6.2 of [5]) two new families of Cremona transformations of bidegree $(2,2)$, one in each dimension $m \geqslant 4$, by specifying the associated base locus schemes that can be :

(A) the union $\Pi_2 \cup \Pi_3 \cup \ell \subset \mathbb{P}^m$ where Π_i is a projective subspace of codimension i for $i = 2, 3$ and where ℓ is a line such that Π_3 and ℓ are disjoint and each intersects Π_2 in dimension $m-4$ and 0 respectively;

(B) the schematic union of a double linear subspace $\mathcal{P}$ of dimension $m-3$ in a hyperplane H with a smooth conic C that is tangent to H at the point $C \cap \mathcal{P}_{\mathrm{red}}$.

By the JC-Correspondence recalled above and proved in [31], we know that, modulo composition by linear automorphisms, quadro-quadric Cremona transformations are nothing but adjoints of rank 3 Jordan algebras. Here, we use this result to describe a general construction of new Cremona transformations of bidegree $(2,2)$ starting from known ones. It seems to be unknown despite its simplicity (in particular, see Section 2.2.1 below).

Let J be a Jordan algebra and M_1, M_2 be two Jordan J-(bi)modules (see [18]): for $i = 1, 2$, M_i is a (non-unital) Jordan algebra and there is a bilinear product $J \times M_i \to M_i : (x, m_i) \mapsto x \cdot m_i$ such that $J_i = J \oplus M_i$ with the product defined explicitly by

$$(x, m_i) \cdot (x', m_i') = (xx', xm_i' + x'm_i + m_i m_i')$$

is a Jordan algebra with unity $(e, 0)$, where e stands for the unity of J.

Then one defines the *'gluing of M_1 and M_2 along J'* as the space

$$J(M_1, M_2) = J \oplus M_1 \oplus M_2$$

endowed with the product $\bullet$ explicitly defined by

$$\begin{aligned}(x, m_1, m_2) \bullet (x', m_1', m_2') \\ = \big(xx', x \cdot m_1' + x' \cdot m_1 + m_1 m_1', x \cdot m_2' + x' \cdot m_2 + m_2 m_2'\big).\end{aligned}$$

It can be verified that this product verifies the Jordan identity and makes of $J(M_1, M_2)$ a Jordan algebra. Its unity is $(e, 0, 0)$ and it contains in a natural way J, J_1 and J_2 as Jordan subalgebras.

Let us remark that if M_3 is a Jordan J-module satisfying the same properties shared by M_1 and M_2 and if J_3 stands for $J \oplus M_3$ with the Jordan product induced by the J-module structure on M_3, then the three Jordan

algebras $J_1(M_2, M_3)$, $J_2(M_1, M_3)$ and $J_3(M_1, M_2)$ identify naturally. We will designate the corresponding algebra by $J(M_1, M_2, M_3)$. Of course, this construction generalizes and can be iterated as many times as wanted.

Now assume that J_1 and J_2 have the same rank r and that there exists a norm (in the sense of [4, II.Section 5]) $N \in \mathrm{Sym}^r(J^*)$ on J such that $N(x) = N_1(x, m_1) = N_2(x, m_2)$ for every $(x, m_1, m_2) \in J \times M_1 \times M_2$, where $N_i \in \mathrm{Sym}^r(J_i^*)$ is the generic norm of J_i for $i = 1, 2$. For every $x \in J$ and $m_i \in M_i$ $(i = 1, 2)$, let $m_i^{\#x}$ be the projection onto M_i of $(x, m_i)^{\#}$. For instance, when $r = 3$, if $T = dN_e \in J^*$ is the trace associated to N, then for $i = 1, 2$, one has:

$$m_i^{\#x} = 2x \cdot m_i + m_i^2 - T(x)m_i.$$

Then one proves easily the following result:

PROPOSITION 2.1. — *Under the assumptions above, the algebra $J(M_1, M_2)$ has rank r and the adjoint is given by*

$$(x, m_1, m_2)^{\#} = \left(x^{\#}, m_1^{\#x}, m_2^{\#x}\right).$$

Let us now explain how this result can be put in practice to construct families of Jordan algebras. For any automorphism $\varphi : J \to J$ of Jordan algebras, let M_2^{φ} be the (non-unital) Jordan algebra M_2 endowed with the new bilinear map $J \times M_2 \to M_2\,, (x, m_2) \mapsto x \cdot_{\varphi} m_2 = \varphi(x) \cdot m_2$. This $\mathbb{C}$-bilinear map makes of M_2^{φ} a Jordan J-module, as one verifies without difficulty.

Assume that J is semi-simple and that $M_1, \ldots, M_k$ are radicial Jordan modules (meaning that $\mathrm{Rad}(J_i) = M_i$ for every i if J_i stands for the Jordan algebra structure on $J \oplus M_i$ induced by the J-module structure of M_i). If $\varphi_1, \ldots, \varphi_m$ are automorphism of J, it follows from the preceding considerations that

$$J(M_1^{\varphi_1}, \ldots, M_m^{\varphi_m})$$

is a Jordan algebra, having the same rank as J. Note that $J(M_1^{\varphi_1}, \ldots, M_m^{\varphi_m})$ and $J(M_1, M_2^{\varphi_2 \circ {\varphi_1}^{-1}}, \ldots, M_m^{\varphi_m \circ {\varphi_1}^{-1}})$ are clearly isomorphic, hence one can assume that $\varphi_1 = \mathrm{I}d$ without any real loss of generality.

Let us consider the preceding construction in the case when $r = 3$, the one we are more interested in. Proposition 2.1 provides a way of constructing new rank 3 Jordan algebras starting from finitely many rank 3 Jordan algebras $J_1, \ldots, J_m$ with isomorphic semi-simple parts, yielding new quadro-quadric Cremona transformations arising from quadro-quadric Cremona transformations with the same semi-simple part (see [31] for definitions).

Let us make explicit this construction in terms of Cremona transformations, which is very simple but quite surprisingly unknown.

For $i = 1, \dots, m$, let F_i be a quadratic affine lift to V_i of a quadro-quadric Cremona transformation $f_i : \mathbb{P}(V_i) \dashrightarrow \mathbb{P}(V_i)$ (with $\dim V_i \geqslant 3$). Assume that the F_i's have the same semi-simple part (see [31, Section 5.1]). In particular there exist complex vector spaces $R_1, \dots, R_m$ and V such that $V_i = V \oplus R_i$ for every $i = 1, \dots, m$. Moreover there are a semi-simple quadro-quadric map $F_{\mathrm{ss}} \in \mathrm{Sym}^2(V^*) \otimes V$ (see [31, Table 2]) and bilinear applications $\mathcal{F}_i : V_i \times R_i \to R_i$ such that for every $i = 1, \dots, m$, the map F_i can be written as

$$F_i\big(\boldsymbol{x}, \boldsymbol{y}_i\big) = \big(F_{\mathrm{ss}}(\boldsymbol{x}), \mathcal{F}_i\big(\boldsymbol{x}, \boldsymbol{y}_i\big)\big) \tag{2.1}$$

in some systems of linear coordinates $\boldsymbol{x} = (x_1, \dots, x_n)$ on V and $\boldsymbol{y}_i = (y_{i1}, \dots, y_{ir_i})$ on R_i (where $r_i = \dim R_i$ for every i)[(5)]. To simplify, we assume that F_{ss} is involutorial so that there exists a cubic form $N(x)$ such that $F_{\mathrm{ss}}(F_{\mathrm{ss}}(x)) = N(x)x$ for every $x \in V$. For $k = 2, \dots, m$, let φ_k be a linear automorphism of V such that $\varphi_k \circ F_{\mathrm{ss}}(x) = F_{\mathrm{ss}} \circ \varphi_k(x)$ and $N(\varphi_k(x)) = N(x)$ for every $x \in V$. Then setting $R = R_1 \oplus \cdots \oplus R_m$, one defines the '*gluing of $f_1, \dots, f_m$ along their semi-simple part by mean of $\varphi_2, \dots, \varphi_m$*' as the rational map

$$f_1 \cdot f_2^{\varphi_2} \cdot \dots \cdot f_{m-1}^{\varphi_{m-1}} \cdot f_m^{\varphi_m} : \mathbb{P}(V \oplus R) \dashrightarrow \mathbb{P}(V \oplus R)$$

which is the projectivization of the affine quadratic map given by the following formula

$$\begin{aligned} &F_1 \cdot F_2^{\varphi_2} \cdot \dots \cdot F_m^{\varphi_m}(\boldsymbol{x}, \boldsymbol{y}) \\ &\qquad = \Big(F_{ss}(\boldsymbol{x}), \mathcal{F}_1\big(\boldsymbol{x}, \boldsymbol{y}_1\big), \mathcal{F}_2\big(\varphi_2(\boldsymbol{x}), \boldsymbol{y}_2\big), \dots, \mathcal{F}_m\big(\varphi_m(\boldsymbol{x}), \boldsymbol{y}_m\big)\Big) \end{aligned}$$

in the linear coordinates $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_m)$ on $V \oplus R$. Using Proposition 2.1, one can prove that

$$f_1 \cdot f_2^{\varphi_2} \cdot \dots \cdot f_m^{\varphi_m} \in \mathbf{Bir}_{2,2}\big(\mathbb{P}(V \oplus R)\big).$$

2.2. Some new families of quadro-quadric Cremona transformations

We use now the method presented above to construct new explicit families of quadro-quadric Cremona transformations in arbitrary dimensions.

(5) To simplify the formula, we have abridged the notation writing $\mathcal{F}_i(\boldsymbol{x}, \boldsymbol{y}_i)$ instead of $\mathcal{F}_i\big((\boldsymbol{x}, \boldsymbol{y}_i), \boldsymbol{y}_i\big)$ in (2.1).

2.2.1. A family of very simple new elements of $\mathbf{Bir}_{2,2}(\mathbb{P}^n)$ $(n \geqslant 2)$

Here we construct new quadro-quadric Cremona transformations that are particularly simple. For $i = 1, 2, 3$, let A_i be a complex vector space of finite dimension $\alpha_i \geqslant 0$ and let A be their direct sum: $A = A_1 \oplus A_2 \oplus A_3$. One sets $\alpha = \dim A = \alpha_1 + \alpha_2 + \alpha_3 \geqslant 0$ and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. Choosing some linear coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ on $\mathbb{C}^3$ and $\boldsymbol{a}_i = (a_{i1}, \dots, a_{i\alpha_i})$ on A_i for $i = 1, 2, 3$, one defines a quadratic map $F^{\underline{\alpha}}$ on $\mathbb{C}^3 \oplus A = \mathbb{C}^{3+\alpha}$ by setting

$$F^{\underline{\alpha}}(\boldsymbol{x}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \Big(x_2x_3, x_1x_3, x_1x_2, x_1\boldsymbol{a}_1, x_2\boldsymbol{a}_2, x_3\boldsymbol{a}_3\Big) \tag{2.2}$$

for every $(\boldsymbol{x}, \boldsymbol{a}) \in \mathbb{C}^3 \oplus A$, with of course $x_i\boldsymbol{a}_i = (x_ia_{i1}, \dots, x_ia_{i\alpha_i})$ for $i = 1, 2, 3$.

One verifies immediatly that $F^{\underline{\alpha}}(F^{\underline{\alpha}}(\boldsymbol{x}, \boldsymbol{a})) = x_1x_2x_3(\boldsymbol{x}, \boldsymbol{a})$ for every $(\boldsymbol{x}, \boldsymbol{a}) \in \mathbb{C}^3 \oplus A$. This implies that the projectivization of $F^{\underline{\alpha}}$ is an involutive quadro-quadric Cremona transformation: for every $\underline{\alpha} \in \mathbb{N}^3$, one has

$$f^{\underline{\alpha}} = [F^{\underline{\alpha}}] \in \mathbf{Bir}_{2,2}(\mathbb{P}^{2+\alpha}).$$

The simplest case when $\underline{\alpha} = (0, 0, 0)$ is well known: $f^{(0,0,0)}$ is nothing but the standard Cremona involution of $\mathbb{P}^2$. The quadro-quadric maps $f^{(m-3,1,0)}$ for $m \geqslant 3$ are the ones considered recently by Bruno and Verra and recalled in (A) at the beginning of Section 2.1 above. Therefore although some particular cases have been considered recently, it seems that the general quadratic involution $f^{\underline{\alpha}}$ considered above has been overlooked although its definition in coordinates (2.2) is certainly one of the simplest that could be imagined.

2.2.2. A family of elements of $\mathbf{Bir}_{2,2}(\mathbb{P}^{2n})$ $(n \geqslant 1)$ constructed from the algebra $\mathbb{C}[\varepsilon]/(\varepsilon^3)$

Let us consider the rank 3 associative algebra $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ as a Jordan algebra, whose adjoint is easily seen to be

$$(a, b, c)^{\#} = (a^2, -ab, b^2 - ac) =: F(a, b, c)$$

in the system of linear coordinates associated to the basis $(1, \varepsilon, \varepsilon^2)$.

For $n \geqslant 1$, let $F_n : \mathbb{C}^{2n+1} \to \mathbb{C}^{2n+1}$ be the quadratic map defined by

$$F_n(a, b_1, c_1, \dots, b_n, c_n) = (a^2, -ab_1, b_1^2 - ac_1, \dots, -ab_n, b_n^2 - ac_n)$$

in some coordinates $(a, b_1, c_1, \dots, b_n, c_n)$ on $\mathbb{C}^{2n+1}$. The map F_n is 'involutorial' in the sense that

$$F_n\big(F_n(a, b_1, \dots, c_n)\big) = a^3(a, b_1, \dots, c_n)$$

for every $(a, b_1, \ldots, c_n) \in \mathbb{C}^{2n+1}$. Then $f_n = [F_n] \in \mathbf{Bir}_{2,2}(\mathbb{P}^{2n})$ for every $n \geqslant 1$.

2.2.3. Construction of continuous families of quadro-quadric Cremona transformations

All the families of quadro-quadric Cremona transformations presented in the two previous subsections are countable. This a consequence of the fact that the semi-simple parts of the afore-mentioned examples have *small* automorphism groups.

In fact, using the general construction of Section 2.1, it is not difficult to construct a continuous family of Cremona transformations of bidegree $(2, 2)$ by starting with a rank 3 Jordan algebra J whose semi-simple part J_{ss} is a rank 3 simple Jordan algebra. Indeed, in this case, the automorphism group $\mathrm{Aut}(J_{\mathrm{ss}})$ of J_{ss} is a simple Lie group of positive dimension. Then if J is an extension of J_{ss} by a non-trivial radical R of dimension $r > 0$, we can construct the family

$$\left\{ J_{\mathrm{ss}}(R, R^{\varphi}) \,\middle|\, \varphi \in \mathrm{Aut}(J_{\mathrm{ss}}) \right\}$$

of rank 3 Jordan algebras of dimension $\dim(J_{\mathrm{ss}}) + 2r$. By considering the associated adjoint maps of these algebras, we get a family of quadro-quadric Cremona transformations of $\mathbb{P}^{\dim(J_{\mathrm{ss}})+2r-1}$ parametrized by the simple Lie group $\mathrm{Aut}(J_{\mathrm{ss}})$.

2.3. From quadro-quadric Cremona transformations to cubo-cubic ones

By definition, a *cubo-cubic Cremona transformation* is a birational map of a projective space of bidegree $(3, 3)$. Such maps have been studied in low dimension (see [17, 27] for instance) but except in dimension 2 and 3 no general classification or structure results are known for them. We indicate below two distinct general constructions of cubo-cubic Cremona maps starting from quadro-quadric ones.

The first one is based on Spampinato's remark recalled in Section 1.

PROPOSITION 2.2. — *Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, $x \mapsto [f_0(x) : \cdots : f_n(x)]$ be a Cremona transformation of bidegree $(2, 2)$. If $N(x) = 0$ is a cubic equation cutting out the secant scheme of the base locus scheme of f (see* [31]*), then*

$$\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}, \quad [x : r] \longmapsto \big[r f_0(x) : \cdots : r f_n(x) : N(x)\big]$$

is a Cremona transformation of bidegree $(3, 3)$ of Spampinato type.

Of course, via the previous construction one can produce from a given $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^n)$, a Cremona transformation on $\mathbb{P}^{n+k}$ of bidegree $(2+k, 2+k)$, for any $k \geqslant 1$. For example from this point of view, the *standard involution* of $\mathbb{P}^{n-1}$ that is the birational map

$$[x_1 : x_2 : \ldots : x_n] \longmapsto \left[x_2x_3 \ldots x_n : x_1x_3 \ldots x_n : \ldots : x_1x_2 \ldots x_{n-1}\right]$$

of bidegree $(n-1, n-1)$, which is the projectivization of the adjoint of the associative and commutative n-dimensional unital algebra $\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$, is obtained from the *ordinary quadratic* transformation of $\mathbb{P}^2$

$$[x_1 : x_2 : x_3] \longmapsto \left[x_2x_3 : x_1x_3 : x_1x_2\right]$$

via Spampinato construction, that is via direct product.

The second construction of a cubo-cubic Cremona transformation from a given $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^n)$ is also classical but less elementary and its generalization to higher degree is not clear at all. It is again based on an algebraic construction(6), but we shall describe it geometrically. The XJC-Correspondence assures that we can associate to $f = [F] \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$ an irreducible projective variety $X_f \subset \mathbb{P}^{2n+1}$, of dimension n, defined as the closure of the image of the affine parametrization $\boldsymbol{x} \mapsto [1 : \boldsymbol{x} : F(\boldsymbol{x}) : N(\boldsymbol{x})]$, notation as in Proposition 2.2, and which is 3-covered by twisted cubic curves according to Proposition 3.3 of [31].

Moreover, in [31, Corollary 5.4]) we remarked that X_f is an OADP-variety: a general point $p \in \mathbb{P}^{2n+1}$ belongs to a unique secant line to X_f, denoted by ℓ_p. For such a point p, there exists a unique unordered pair $\{a_p, b_p\}$ of two distinct points of X_f such that $\ell_p = \langle a_p, b_p \rangle$. Thus one can define p' as the projective harmonic conjugate of the triple (a_p, b_p, p) on the

(6) Let J be a rank 3 Jordan algebra with trace $T(x)$ and cubic norm $N(x)$. By [1, Section 8.v], the space of *Zorn matrices with coefficients in* J defined by

$$Z_2(J) = \left\{ \begin{pmatrix} a & x \\ y & b \end{pmatrix} \;\middle|\; \begin{array}{c} a, b \in \mathbb{C} \\ x, y \in J \end{array} \right\},$$

together with the product $\bullet$ and the involution $M \mapsto \overline{M}$ given respectively by

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \bullet \begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + T(x, y') & ax' + b'x + y\#y' \\ a'y + by' + x\#x' & bb' + T(x', y) \end{pmatrix} \text{ and } \overline{\begin{pmatrix} a & x \\ y & b \end{pmatrix}} = \begin{pmatrix} b & x \\ y & a \end{pmatrix}$$

(where $x\#y = (x+y)^\# - x^\# - y^\#$ for every $x, y \in J$) is a *structurable algebra*, meaning that the triple product $[M, N, P] = (M \bullet \overline{N}) \bullet P + (P \bullet \overline{N}) \bullet M - (P \bullet \overline{M}) \bullet N$ satisfies some particular algebraic identities. Moreover, the subspace $\{M \in Z_2(J) | \overline{M} = -M\}$ of skew-elements is 1-dimensional and spanned by the diagonal matrix, noted by $\boldsymbol{\sigma}$, with scalar diagonal coefficients 1 and -1. Then it can be proved (see formula (1.5) in [2]) that the projectivization of the map $M \mapsto \boldsymbol{\sigma} \bullet [M, \boldsymbol{\sigma} \bullet M, M]$ is an involutive cubo-cubic Cremona transformation of $\mathbb{P}Z_2(J)$.

projective line ℓ_p: p' is the unique point on ℓ_p such that $\mathrm{Cr}(a_p, b_p; p, p') = -1$[(7)].

PROPOSITION 2.3. — *For any $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$, $n \geqslant 3$, the map*

$$\Phi_f : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{2n+1}$$
$$p \longmapsto p'$$

defined above is an involutorial cubo-cubic Cremona transformation.

Proof. — From the fact that for every $q \in \ell_p \setminus \{a_p, b_p\}$ the line ℓ_p is the unique secant lines to X_f passing through q and from well-known properties of the cross-ratio it follows that $\Phi_f \circ \Phi_f = \mathrm{Id}$ as a rational map so that Φ_f is a birational involution of $\mathbb{P}^{2n+1}$.

To prove that Φ_f is cubo-cubic, one can relate it with the cubic map considered in Footnote 6 by verifying that the arguments and formulae of [19], that concern a priori only the semi-simple case, are in fact valid in full generality.

A more geometrical approach is the following one: for $p \in \mathbb{P}^{2n+1}$ general, the points a_p and b_p are general points of X_f. In particular, since X_f is 3-covered by twisted cubics, there exist twisted cubics curves included in X_f and passing through a_p and b_p. Let C_p be such a curve. Since C_p is a twisted cubic, it is an OADP variety in its span $\langle C_p \rangle \simeq \mathbb{P}^3$ and Φ_f induces a Cremona involution $\Phi_{C_p} : \langle C_p \rangle \dashrightarrow \langle C_p \rangle$. Since Φ_{C_p} is an involution of bidegree $(3,3)$ in $\mathbb{P}^3$, we deduce that Φ_f is of the same type, concluding the proof. □

Since $X_f \subset \mathbb{P}^{2n+1}$ is an OADP-variety, the projection π_x from a general tangent space $T_x X$ induces a birational map $\pi_x : X_f \dashrightarrow \mathbb{P}^n$. Thus, for $x_1, x_2 \in X_f$ general points, the birational map $\pi_{x_1}^{-1} \circ \pi_{x_2} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is easily seen to be a Cremona transformation of *Spampintato type* linear equivalent to the map considered in Proposition 2.2, see also [31, Section 3.2.2].

3. An algebraic approach to the description of $\mathbf{Bir}_{2,2}(\mathbb{P}^n)$ for n small

Via the *JC*-correspondence one can classify quadro-quadric Cremona transformations of $\mathbb{P}^{n-1}$ by using the classification of n-dimensional rank 3 Jordan algebras. After the pioneering work by Albert and Jacobson (among

(7) Here $\mathrm{Cr}(\cdot,\cdot;\cdot,\cdot)$ stands for the cross-ratio of two pairs of points on $\ell_p \simeq \mathbb{P}^1$.

many others) in the 50's, we have now a wide range of particularly powerful algebraic tools to study Jordan algebras. In what follows, we will shortly present some notions and results to be applied to determine rank 3 Jordan algebras in low dimension. In particular we shall obtain the complete classification of rank 3 Jordan algebras of dimension 5, from which we will deduce a complete list of involutorial normal forms for quadro-quadric Cremona transformations of $\mathbb{P}^4$.

3.1. Some classical notions and tools in the theory of Jordan algebras

The following material is very classical and is presented in most of the standard references on Jordan algebras (as [18, 33, 4, 25] for instance).

3.1.1. Nilpotent and nil algebras

Let A be a complex algebra, only assumed to be commutative, but not necessarily Jordan or with a unity. If A_1, A_2 are two subsets of A, one sets $A_1A_2 = A_2A_1 = \{a_1a_2 \mid a_i \in A_i,\, i = 1, 2\}$. Then one defines inductively A^k for $k \geqslant 1$ by setting $A^1 = A$ and $A^k = \cup_{0<p<k} A^p A^{k-p}$ for $k > 1$. The algebra A is said to be *k-nilpotent* if $A^k = 0$ but $A^{k-1} \neq 0$ and it is said *nilpotent* if it is k-nilpotent for a certain $k \geqslant 1$. By definition, A is a *nil algebra* if it is a *k-nil algebra* for a certain positive integer k, i.e. if $a^k = 0$ for every $a \in A$.

Now let R be the nontrivial radical of a Jordan algebra or rank ρ. Then it can be proved that $r^\rho = 0$ for every $r \in R$ so that R is a k-nil algebra for a certain $k \in \{2, \ldots, \rho\}$. In fact, much more is true since Albert proved that R is nilpotent and not only nil, a result not used in the sequel.

3.1.2. Peirce decomposition

By definition, an *idempotent* of a Jordan algebra J is a nonzero element $u \in J$ verifying $u^2 = u$. For instance, the unity e of J is idempotent. Let u be a fixed idempotent. One proves that the multiplication L_u by u satisfies the relation $L_u(L_u - I_d)(2L_u - I_d) = 0$, yielding the direct sum decomposition $J = J_0(u) \oplus J_1(u) \oplus J_{\frac{1}{2}}(u)$, where $J_\lambda(u) = \mathrm{Ker}(L_u - \lambda\, I_d) = \{v \in J \mid u \cdot v = \lambda\, v\}$ for $\lambda \in \{0, 1, 1/2\}$.

Two elements $u_1, u_2 \in J$ are *orthogonal* if $u_1 \cdot u_2 = 0$. In this case, if both are idempotents, then their sum $u_1 + u_2$ is idempotent too. An idempotent u is *irreducible* when it cannot be written $u = u_1 + u_2$ where u_1, u_2 are two orthogonal idempotents. Since J has finite dimension, one verifies easily that any idempotent u admits an *irreducible decomposition by orthogonal idempotents*, that is can be written as $u = u_1 + \cdots + u_m$ where $u_1, \ldots, u_m$ are pairwise orthogonal irreducible idempotents (then m is well-defined and depends only on u).

Let $e = e_1 + \cdots + e_m$ be such a decomposition for the unity e of J. Then for $i, j = 1, \ldots, m$ distinct, one sets $J_{ii} = J_1(e_i) = \{x \in J \mid e_i \cdot x = x\}$ and $J_{ij} = J_{\frac{1}{2}}(e_i) \cap J_{\frac{1}{2}}(e_j) = \{x \in J \mid e_i \cdot x = e_j \cdot x = \frac{1}{2}x\}$.

Proposition 3.1. — *There is a direct decomposition*

$$J = \bigoplus_{1 \leqslant i \leqslant j \leqslant m} J_{ij}. \tag{3.1}$$

Moreover, for every distinct $i, j, k, \ell \in \{1, \ldots, m\}$, one has

$$(J_{ii})^2 \subset J_{ii}\,, \quad J_{ii} \cdot J_{ij} \subset J_{ij}\,, \quad (J_{ij})^2 \subset J_{ii} \oplus J_{jj}\,, \quad J_{ij} \cdot J_{jk} \subset J_{ik} \tag{3.2}$$

and

$$J_{ii} \cdot J_{jj} = J_{ij} \cdot J_{kk} = J_{ij} \cdot J_{k\ell} = 0. \tag{3.3}$$

Finally, setting $R = \mathrm{Rad}(J)$, one has

$$R = \Big(\bigoplus_{i=1}^{m} \mathrm{Rad}(J_{ii}) \Big) \oplus \Big(R \cap \big(\bigoplus_{i<j} J_{ij} \big) \Big). \tag{3.4}$$

At least theoretically, the material presented in this subsection should be sufficient to classify rank 3 Jordan algebras of 'reasonable dimension'. More rigorously, the notions just introduced above reduce the classification of Jordan algebras to some problems in linear algebra, which although simple from a conceptual point of view, can become immediately quite complicated as soon as the dimension increases.

Let us mention here the reference [40] where H. Wesseler presents the classification of complex Jordan algebras of dimension less or equal to 6. The first named author of the present paper has written a text [30] based on [40] and providing complete proofs of the classification of rank 3 Jordan algebra of dimension less or equal to 6. Since the report [40] is not well-known, has not been published (and it will not be published), we decided to present in the next subsection some details on the classification of Jordan algebras in dimension 4 and 5, following very closely Wesseler's approach and arguments.

3.1.3. Classification of nil Jordan algebras of low dimension

A Jordan algebra with radical of codimension 1 is isomorphic to the unitalization of its radical. Then to classify rank 3 Jordan algebras one needs in particular to know all the nil Jordan algebras of nilindex less or equal to 3. We recall below the classification of these in low dimension (for further references on this, the reader can consult [14, 12, 21]).

PROPOSITION 3.2. — *A nontrivial nilalgebra of nilindex at most 3 and of dimension n less or equal to 4 is isomorphic to an algebra with basis $(v_1, \dots, v_n)$ in the following table:*

Dimension	**Algebra**	**Non trivial products**
2	$\mathcal{R}^2$	$v_1^2 = v_2$
3	$\mathcal{R}_1^3$	$v_1^2 = v_3$
3	$\mathcal{R}_2^3$	$v_1^2 = v_2^2 = v_3$
4	$\mathcal{R}_1^4$	$v_1^2 = v_3$, $v_1 v_2 = v_4$
4	$\mathcal{R}_2^4$	$v_1^2 = v_2^2 = v_3$, $v_1 v_2 = v_4$
4	$\mathcal{R}_3^4$	$v_1^2 = v_4$
4	$\mathcal{R}_4^4$	$v_1^2 = v_2^2 = v_4$
4	$\mathcal{R}_5^4$	$v_1^2 = v_2^2 = v_3^2 = v_4$

Table 1. Classification of nilalgebras of nilindex $\leqslant 3$ in low dimensions.

If $\dim(R) = 2$, then the conclusion is trivial. Let us prove the above classification for $\dim R = 4$, letting the remaining cases to the reader. First of all we recall some useful results of [14]: since R is nilpotent and non trivial, one has $R^2 \subsetneq R$, implying $\dim R^2 \in \{1, 2, 3\}$ ($R^2 = 0$ must be excluded since R is not trivial by hypothesis). Assume that $\dim R^2 = 3$. Then there exists $r \in R$ such that $R = \mathbb{C}r \oplus R^2$. Then $R^2 = \mathbb{C}r^2 + rR^2 + (R^2)^2 \subset \mathbb{C}r^2 + R^3$, yielding $R^2 = \mathbb{C}r^2 + R^3$. This implies $R = \mathrm{Span}\{r, r^2\} + R^3$. By repeating these arguments, one proves that $R = \mathrm{Span}\{r, \dots, r^{k-1}\} + R^k$ for every $k \geqslant 1$. Since $r^3 = 0$ and since R is nilpotent, we would deduce $R = \mathrm{Span}\{r, r^2\}$ and $\dim(R) = 2$, contrary to our assumption. Thus necessarily $\dim R^2 \in \{1, 2\}$.

Assume first $\dim(R^2) = 2$. Let $r, r' \in R$ such that $r^2$and rr' span R^2. Then $(r')^2 = \alpha\, r^2 + \beta\, rr'$ with $\alpha, \beta \in \mathbb{C}$. By replacing r' by $r' - \beta r/2$,

we can suppose $(r')^2 = \alpha\, r^2$. If $\alpha \neq 0$, replacing r by $r/\sqrt{\alpha}$, we can also suppose $\alpha = 1$. Then one obtains two cases: the algebras $\mathcal{R}_1^4$ and $\mathcal{R}_2^4$ in the table above.

Assume now $\dim(R^2) = 1$. Let r_1 be such that $R^2 = \mathbb{C}\, r_4$ where $r_4 = (r_1)^2$ and choose r_2 and r_3 such that $(r_1, \ldots, r_4)$ is a basis of R. The product on R is determined by the quadratic form φ on $R' = \mathrm{Span}\{r_1, r_2, r_3\}$ defined by the relation $rr' = \varphi(r, r')r_4$ for $r, r' \in R'$. Note that φ is non-trivial since $\varphi(r_1) = 1$. Moreover, one verifies that isomorphic quadratic forms on R' induce isomorphic Jordan algebras. Hence there are only three possibilities corresponding to the possible values 1, 2 or 3 for the rank of the quadratic form. The corresponding algebras are denoted by $\mathcal{R}_3^4, \mathcal{R}_4^4$ and $\mathcal{R}_5^4$ in TABLE 1 above.

3.2. Rank 3 Jordan algebras of dimension 3 and quadro-quadric Cremona transformations of $\mathbb{P}^2$

It is an easy exercise to determine all Jordan algebras of dimension 3 by using the *JC*-correspondence: a quadro-quadric Cremona transformation of the projective plane is given by its base locus scheme that is a non-degenerate 0-dimensional subscheme of length 3 in $\mathbb{P}^2$. One sees immediately that there are exactly three such subschemes (up to isomorphisms) and that they belong to the same irreducible component of $\mathrm{Hilb}^3(\mathbb{P}^2)$. Consequently, up to isomorphisms, there are three rank 3 Jordan algebras of dimension 3.

The classification is collected in the following table:

Algebra	**dim(R)**	**Semi-simple part**	**Adjoint $(x, y, z)^\#$**	**Base locus**
$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	0	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	(yz, xz, xy)	
$\mathbb{C} \times \frac{\mathbb{C}[\varepsilon]}{(\varepsilon^2)}$	1	$\mathbb{C} \times \mathbb{C}$	$(y^2, xy, -xz)$	
$\frac{\mathbb{C}[\varepsilon]}{(\varepsilon^3)}$	2	$\mathbb{C}$	$(x^2, -xy, y^2 - xz)$	

Table 2. Classification of rank 3 Jordan algebras of dimension three or equivalently, of quadro-quadric Cremona transformations of $\mathbb{P}^2$.

3.3. Rank 3 Jordan algebras of dimension 4 and quadro-quadric Cremona transformations of $\mathbb{P}^3$

The classification of bidegree $(2,2)$ Cremona transformations in $\mathbb{P}^3$ has been considered recently in [28] and classically by Enriques and Conforto (see all the references in [28]).

In this section, we give the classification of rank 3 Jordan algebras in dimension 4. This classification is classical too and for this reason some cases will be left to the reader. Recent references on the subject are [40, 20, 30]. Concerning the more particular case of associative algebras, one can consult the classical papers [39, 34] or the more recent one [13] (see also the references therein).

The classification given by TABLE 3 below is easy to obtain. Let J be a rank 3 Jordan algebra of dimension 4. First of all, when $R = \mathrm{Rad}(J)$ is trivial, J is semi-simple, hence it is the direct product of $\mathbb{C}$ with $\mathcal{J}^3_{Q,2}$.

When $\dim(R) = 1$, the semi-simple part J_{ss} of J has dimension 3. It cannot be of rank 1 (it has dimension 3) or of rank 2: in the latter case, it would be isomorphic to $\mathcal{J}^3_{Q,2}$ that does not admit a cubic norm. Hence J_{ss} has rank 3 and can be assumed to be the direct product of three copies of $\mathbb{C}$. Clearly $R^2 = 0$ hence the Jordan product is completely determined by the three complex numbers α_1, α_2 and α_3 such that $e_i r = \alpha_i r$ for $i = 1, 2, 3$, where (e_1, e_2, e_3) stands for the standard basis of $J_{\mathrm{ss}} = \mathbb{C}\times\mathbb{C}\times\mathbb{C}$ and where r is a non trivial element of R. It is easy to prove that there exists only one possibility (up to isomorphisms) for the α_i's, namely $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 0$. One easily verifies that the corresponding algebra is isomorphic to $\mathcal{J}^4_2 = \mathbb{C} \times \mathcal{J}^3_{Q,1}$.

We now assume $\dim(R) = 2$. In this case the semi-simple part is $J_{\mathrm{ss}} = \mathbb{C}\times\mathbb{C}$. Let (e_1, e_2) stands for the image of the standard basis of $\mathbb{C}^2 = J_{\mathrm{ss}}$ in an embedding $J_{\mathrm{ss}} \hookrightarrow J$. This is a set of pairwise irreducible idempotents. Let $J = J_{11} \oplus J_{12} \oplus J_{22}$ be the associated Peirce decomposition.

If $J_{12} \cap R = (0)$ there are two possibilities: either (a) $R \subset J_{22}$, or (b) $R = \mathrm{Rad}(J_{11}) \oplus \mathrm{Rad}(J_{22})$ with $\dim \mathrm{Rad}(J_{ii}) \neq 0$ for $i = 1, 2$. In any case, one has $J_{12} = (0)$ and J is isomorphic to the direct product $J_{11} \times J_{22}$ by (3.3). If $R \subset J_{22}$, then J_{11} is isomorphic to $\mathbb{C}$ for dimensional reasons and J_{22} is a Jordan algebra of rank 2, dimension 3 with radical of dimension 2. Hence $J_{22} \simeq \mathcal{J}^3_{Q,0}$ and J is isomorphic to the algebra $\mathcal{J}^4_3 = \mathbb{C} \times \mathcal{J}^3_{Q,0}$. Case (b) does not occur. Indeed, in this case J_{ii} had dimension 2 and rank 2 for $i = 1, 2$ so that $J \simeq J_{11} \times J_{22}$ would have rank 4, contradicting our assumption.

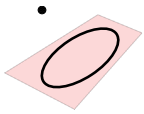
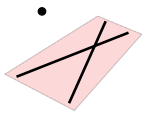
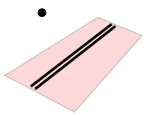
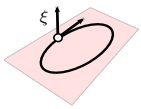
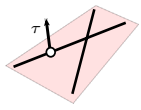
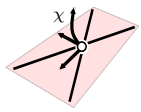
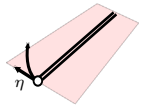
Algebra J	$\dim R$	J_{ss}	Adjoint $(x,y,z,t)^{\#}$	Base locus
$\mathcal{J}_1^4 = \mathbb{C} \times \mathcal{J}_{Q,2}^3$	0	$\mathbb{C} \times \mathcal{J}_{Q,2}^3$	$\left(y^2+z^2+t^2\,,\, xy\,,\, -xz\,,\, -xt\right)$	
$\mathcal{J}_2^4 = \mathbb{C} \times \mathcal{J}_{Q,1}^3$	1	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\left(y^2+z^2\,,\, xy\,,-xz\,,\, -xt\right)$	
$\mathcal{J}_3^4 = \mathbb{C} \times \mathcal{J}_{Q,0}^3$	2	$\mathbb{C} \times \mathbb{C}$	$\left(y^2\,,\, xy\,,-xz\,,-xt\right)$	
$\mathcal{J}_4^4$	2	$\mathbb{C} \times \mathbb{C}$	$\left(xy\,,\, x^2\,,\, t^2-yz\,,\, xt\right)$	
$\mathcal{J}_5^4$	2	$\mathbb{C} \times \mathbb{C}$	$\left(xy, x^2, yz, xt\right)$	
$\mathcal{J}_6^4$	3	$\mathbb{C}$	$\left(x^2\,,-xy\,,-xz\,,\, 2yz-xt\right)$	
$\mathcal{J}_7^4$	3	$\mathbb{C}$	$\left(x^2\,,-xy\,,-xz\,,\, y^2-xt\right)$	

Table 3. Classification of rank 3 Jordan algebras of dimension four or equivalently, of quadro-quadric Cremona transformations of $\mathbb{P}^3$.

We now treat the case $\dim(J_{12} \cap R) = 1$. Let a be such that $R \cap J_{12} = \mathbb{C}a$. By (3.4), one can assume that $\dim \mathrm{Rad}(J_{11}) = 1$ and $\dim \mathrm{Rad}(J_{22}) = 0$. Let b such that $\mathrm{Rad}(J_{11}) = \mathbb{C}b$. For dimensional reasons, it follows that $J_{11} = \mathbb{C}e_1 \oplus \mathbb{C}b$, $J_{12} = \mathbb{C}a$ and $J_{22} = \mathbb{C}e_2$. Since $b^2 \in R \cap J_{11} = \mathbb{C}b$ and because $b^3 = 0$ (recall that J has rank 3 and that $b \in \mathrm{Rad}(J)$) we get $b^2 = 0$. The structure of J will be completely determined when the two products

ab and a^2 are. According to (3.2), one has $ab \in J_{12} \cdot J_{11} \subset J_{12} = \mathbb{C}a$, hence there exists $\kappa_1 \in \mathbb{C}$ such that $ab = \kappa_1 a$. Hence $L_b(a) = \kappa_1 a$, thus $\kappa_1 = 0$ because the multiplication L_b by b is also nilpotent, as one easily sees. Finally, one has $a^2 \in R \cap J_{12}^2 \subset R \cap (J_{11} \oplus J_{22}) = \mathbb{C}b$ so that there exists $\kappa \in \mathbb{C}$ such that $a^2 = \kappa b$. In the coordinate system associated to the basis (e_1, b, a, e_2) of J, the Jordan product is given by :

$$\begin{aligned}(x_1, \beta, \alpha, x_2) \cdot (x_1', \beta', \alpha', x_2') \\ = \Big(x_1 x_1', \beta x_1' + x_1 \beta' + \kappa\alpha\alpha', \frac{1}{2}\big((x_1 + x_2)\alpha' + \alpha(x_1' + x_2')\big), x_2 x_2'\Big).\end{aligned}$$

One verifies that the corresponding Jordan algebras have rank 3 and that for any $\kappa \in \mathbb{C}$, the adjoint is given by

$$(x_1, \beta, \alpha, x_2)^{\#} = \big(x_1 x_2, \kappa\alpha^2 - \beta x_2, -\alpha x_1, x_1^2\big).$$

Up to isomorphisms, there are only two cases to be considered, namely $\kappa = 1$ and $\kappa = 0$. We denote respectively by $\mathcal{J}_4^4$ and $\mathcal{J}_5^4$ the two corresponding Jordan algebras.

Finally, when $\dim(R) = 3$ the multiplicative structure of J is completely determined by that of R. Thus there are two possibilities in this case, denoted by $\mathcal{J}_6^4$ and $\mathcal{J}_7^4$ in the table above, that correspond respectively to the unitalizations of the nilalgebras $\mathcal{R}_2^3$ and $\mathcal{R}_1^3$ of TABLE 1.

3.4. Quadro-quadric Cremona transformations of $\mathbb{P}^4$ and rank 3 Jordan algebras of dimension 5

In this section, we use the same strategy to obtain the main result of this paper: a complete and explicit classification of quadro-quadric Cremona transformations of $\mathbb{P}^4$.

3.4.1. The three generic quadro-quadric Cremona transformations of $\mathbb{P}^4$

In [5], Bruno and Verra give a modern proof of the following result firstly proved by Semple in [35], providing the classification of the base locus schemes of general elements of $\mathbf{Bir}_{2,2}(\mathbb{P}^4)$.

THEOREM 3.3. — *The base locus scheme of a general quadro-quadric Cremona transformation of $\mathbb{P}^4$ is projectively equivalent to one of the subschemes $\mathcal{B}_I, \mathcal{B}_{II}$ an $\mathcal{B}_{III}$ of $\mathbb{P}^4$ where:*

(I) *$\mathcal{B}_I$ is the disjoint union of a smooth quadric surface Q with a point p lying outside the hyperplane $\langle Q\rangle$;*

(II) *$\mathcal{B}_{II}$ is the union of a 2-plane π with two skew lines ℓ_1 and ℓ_2, each one intersecting π in one point;*

(III) *$\mathcal{B}_{III}$ is the scheme theoretic union of a double line $\mathcal{L}$ in a hyperplane H with a smooth conic C tangent to H at the point $C \cap \mathcal{L}_{\mathrm{red}}$.*

Let us recall that the type T_f of a Cremona transformation $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is (the specification of) the irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ containing $\mathcal{B}_f$. The proof of the previous result also implies that there are exactly three types of elements in $\mathbf{Bir}_{22}(\mathbb{P}^4)$, which will be denoted by I, II and III. Moreover, for any type $T \in \{I, II, III\}$, two general elements in the irreducible component of $\mathrm{Hilb}(\mathbb{P}^4)$ containing $\mathcal{B}_T$ are projectively equivalent. This implies that to T corresponds what we call the *generic Cremona transformation* $f_T \in \mathbf{Bir}_{22}(\mathbb{P}^4)$. Normal forms for f_I, f_{II} and f_{III} as well as the corresponding multidegrees are given in the table below.

Type T	**Base locus $\mathcal{B}_T$**	**Cremona involution $f_T(x,y,z,t,u)$**	**Multidegree**
I	$Q \sqcup \{p\}$	$\big(y^2+z^2+t^2+u^2, xy, -xz, -xt, -xu\big)$	$(2,2,2)$
II	$\pi \cup \ell_1 \cup \ell_2$	$(yz\,,\, xz\,,\, xy\,,\, -zt\,,\, -yu)$	$(2,3,2)$
III	$\mathcal{L} \cup C$	$(xy, x^2, -yz+u^2, -yt, -xu)$	$(2,4,2)$

Table 4. The three generic quadro-quadric Cremona transformations of $\mathbb{P}^4$.

The method used by Semple (and later independently by Bruno and Verra) uses induction on the dimension and can be roughly described as follows: given a point $o \in \mathbb{P}^m$ where a given $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^m)$ is an isomorphism, one takes a general quadric Q passing through o and belonging to the homaloidal linear system $f^{-1}|\mathcal{O}_{\mathbb{P}^m}(1)|$. Then $P = f(Q)$ is a general hyperplane through $f(o)$ and if $\pi : Q \dashrightarrow \mathbb{P}^{m-1}$ stands for the restriction to Q of the linear projection from o, one proves that $h = \pi \circ (f^{-1}|_P) : P \dashrightarrow \mathbb{P}^{m-1}$ is birational, of bidegree $(2,n)$ with $n \in \{2,3,4\}$, and that its base locus scheme $\mathcal{B}_h$ is the union of $f(o)$ with the intersection of $P \simeq \mathbb{P}^{m-1}$ with the base locus scheme $\mathcal{B}_{f^{-1}}$ of f^{-1} (cf. [5, Section 2] for more details).

As remarked by Bruno and Verra, the base locus $\mathcal{B}_h$ of a Cremona map h obtained by Semple's construction described above cannot be too special since it contains $f(o)$ as an isolated point. As we shall see below and as it was already shown by the classification of $\mathbf{Bir}_{2,2}(\mathbb{P}^3)$, a lot of interesting examples appear by degenerating the isolated point to an infinitely near point of the support of the general base locus scheme. Moreover, there is no general description of quadratic Cremona transformations of degree $(2,3)$ and $(2,4)$ in dimension greater than 5 so that Semple's method can work effectively only in dimension at most 4. However, Semple's approach yields quite easily that any quadro-quadric Cremona transformations of $\mathbb{P}^4$ is a degeneration of one of the f_T's in the table above, although it does not say anything precise on the possible degenerations. In other terms, Semple's inductive method does not allow to obtain the complete lists of Cremona transformations of $\mathbb{P}^n$ for $n \geqslant 4$.

On the contrary the classification of 5-dimensional Jordan algebras of rank 3 is not difficult. By using the material of Section 3.1, it amounts to elementary but a bit lengthy exercises in linear algebra.

3.4.2. On the classification of rank 3 Jordan algebras of dimension 5

We now classify Jordan algebras J of rank 3 and of dimension 5. We shall consider the different subcases according to the possible dimension of the radical R of J.

Let us begin with the case $\dim(R) = 4$. Then J is the unitalization of one of the five nilalgebras $\mathcal{R}_1^4, \ldots, \mathcal{R}_5^4$ in TABLE 1 (the case when $R^2 = 0$ would imply that J has rank 2 hence it has to be excluded). For $i = 1, \ldots, 5$, let us denote by $\mathcal{J}_i^5$ the algebra with $\mathcal{R}_i^4$ as radical. The $\mathcal{J}_i^5$'s are associative algebras of rank 3. Explicit expressions for the adjoints of these algebras in the coordinate system (x, y, z, t, u) associated to the basis $(e, v_1, \ldots, v_4)$ are given in TABLE 6 below.

Let us now consider the case $\dim(R) = 3$. This case is not more complicated than the other ones but requires several pages of elementary arguments of linear algebra that are outlined in [30, Section 5.0.4]. Since these details do not present any real conceptual interest, we have decided to exclude them and to state the corresponding results. Let us define $\mathcal{J}_6^5, \ldots, \mathcal{J}_{12}^5$ as the algebras whose multiplicative tables in a certain basis denoted by (e_1, e_2, a, b, d) are given in TABLE 5 below. The algebras $\mathcal{J}_k^5$'s for $k \in \{6, \ldots, 12\}$ are rank 3 Jordan algebras with 3-dimensional radicals.

Moreover, any Jordan algebra of this type is isomorphic to exactly one of the seven following algebras:

$\mathcal{J}_6^5$	e_1	e_2	a	b	d
e_1	e_1		a	$\frac{1}{2}b$	$\frac{1}{2}d$
e_2		e_2		$\frac{1}{2}b$	$\frac{1}{2}d$
a	a				
b	$\frac{1}{2}b$	$\frac{1}{2}b$			
d	$\frac{1}{2}d$	$\frac{1}{2}d$			

$\mathcal{J}_7^5$	e_1	e_2	a	b	d
e_1	e_1		a	$\frac{1}{2}b$	$\frac{1}{2}d$
e_2		e_2		$\frac{1}{2}b$	$\frac{1}{2}d$
a	a				
b	$\frac{1}{2}b$	$\frac{1}{2}b$		a	
d	$\frac{1}{2}d$	$\frac{1}{2}d$			

$\mathcal{J}_8^5$	e_1	e_2	a	b	d
e_1	e_1		a	$\frac{1}{2}b$	$\frac{1}{2}d$
e_2		e_2		$\frac{1}{2}b$	$\frac{1}{2}d$
a	a				
b	$\frac{1}{2}b$	$\frac{1}{2}b$		a	
d	$\frac{1}{2}d$	$\frac{1}{2}d$			a

$\mathcal{J}_9^5$	e_1	e_2	a	b	d
e_1	e_1		a	$\frac{1}{2}b$	$\frac{1}{2}d$
e_2		e_2		$\frac{1}{2}b$	$\frac{1}{2}d$
a	a			d	
b	$\frac{1}{2}b$	$\frac{1}{2}b$	d		
d	$\frac{1}{2}d$	$\frac{1}{2}d$			

$\mathcal{J}_{10}^5$	e_1	e_2	a	b	d
e_1	e_1		a	b	$\frac{1}{2}d$
e_2		e_2			$\frac{1}{2}d$
a	a				
b	b				
d	$\frac{1}{2}d$	$\frac{1}{2}d$			

$\mathcal{J}_{11}^5$	e_1	e_2	a	b	d
e_1	e_1		a	b	$\frac{1}{2}d$
e_2		e_2			$\frac{1}{2}d$
a	a				
b	b				
d	$\frac{1}{2}d$	$\frac{1}{2}d$			a

$\mathcal{J}_{12}^5$	e_1	e_2	a	b	d
e_1	e_1		a	b	d
e_2		e_2			
a	a				
b	b				
d	d				

Table 5. Multiplication tables for 5-dimensional rank 3 Jordan algebras with 3-dimensional radical (where an empty entry means that the corresponding product is equal to zero).

Explicit expressions in the coordinate system (x, y, z, t, u) associated to the basis (e_1, e_2, a, b, d) for the adjoints of the seven algebras $\mathcal{J}_6^5, \ldots, \mathcal{J}_{12}^5$ are given in TABLE 6 below.

Let us assume now $\dim(R) = 2$. In this case, $J_{\rm ss}$ is a semi-simple Jordan algebra of dimension 3 and of rank 2 or 3. Since it admits a cubic norm, it is necessarily of rank 3 and one can assume that $J_{\rm ss} = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. The standard basis (e_1, e_2, e_3) of $J_{\rm ss} = \mathbb{C}^3$ gives an irreducible decomposition $e = e_1 + e_2 + e_3$ of the unity by pairwise primitive orthogonal idempotents. Let us consider the corresponding Peirce decomposition (3.1) and discuss each case according to the value of

$$\theta = \dim \big(R \cap \sum_{i<j} J_{ij}\big) \in \{0, 1, 2\}.$$

If $\theta = 0$, then $J = \oplus_{i=1}^3 J_{ii}$, so it follows from Proposition 3.1 that J is the direct product of the J_{ii}'s. Since at least one of these three Jordan algebras has dimension > 1, it follows that the rank of J is at least $1 + 1 + 2 = 4$, contradicting the assumption $\mathrm{rk}(J) = 3$. Thus the case when $\theta = 0$ does not occur.

Assume now that $\theta = 1$. One can suppose that $R = \mathrm{Rad}(J_{ii}) \oplus J_{23}$ for a certain $i \in \{1, 2, 3\}$, with $\mathrm{Rad}(J_{ii})$ of dimension 1. If $i \in \{2, 3\}$, say $i = 3$, then set $J' = J_{22} \oplus J_{23} \oplus J_{33}$. It follows from Proposition 3.1 that J' is a Jordan algebra since $J_{11} = \mathbb{C}\, e_1$. Moreover, from $J' \cdot J_{11} = 0$, it comes that $J = J_{11} \times J'$ for dimensional reasons. Here J' is a rank 2 Jordan algebra of dimension 4 with a 2-dimensional radical. It can be proved that with respect to a suitable basis, the product of J' is given by $(y, z, t, u) \cdot (y', z', t', u') = (yy' + zz', yz' + zy', yt' + ty', yu' + uy')$. Moreover, one has $e_2 = \frac{1}{2}(1, 1, 0, 0)$ and $e_3 = \frac{1}{2}(1, -1, 0, 0)$ in the corresponding coordinates. Then easy computations show that $J'(e_j) = J_{jj} = \mathbb{C}\, e_j$ for $j = 2, 3$, contradicting the assumption $\dim(\mathrm{Rad}(J_{33})) = 1$. Thus this case does not occur.

Let us now assume that $i = 1$, *i.e.* that $\dim \mathrm{Rad}(J_{ii}) = 1$. As above, one proves that $J = J_{11} \times J'$, but now with $\dim J_{11} = \dim J' = 2$. This would imply that J has rank $2 + 2 = 4$, excluding also this case.

We now treat the case $\theta = 2$. Then one has $\mathrm{Rad}(J_{ii}) = 0$ for $i = 1, 2, 3$ so that $R = J_{12} \oplus J_{13} \oplus J_{23}$. Assume first that no J_{ij}'s (for $i < j$) has dimension 2. Thus one can assume that $J_{23} = 0$ and that J_{12} and J_{13} are 1-dimensional. Then there exist a, b such that $J_{12} = \mathbb{C}\, a$, $J_{13} = \mathbb{C}\, b$. Since $R = J_{12} \oplus J_{13}$, using Proposition 3.1, one deduces easily that the

multiplication table of J is the following:

	e_1	e_2	e_3	a	b
e_1	e_1			$\frac{1}{2}a$	$\frac{1}{2}b$
e_2		e_2		$\frac{1}{2}a$	
e_3			e_3		$\frac{1}{2}b$
a	$\frac{1}{2}a$	$\frac{1}{2}a$			
b	$\frac{1}{2}b$		$\frac{1}{2}b$		

This is the multiplication table of a rank 3 Jordan algebra that will be denoted by $\mathcal{J}^5_{13}$. The expression of the adjoint in the coordinate system associated to the basis (e_1, e_2, e_3, a, b) of $\mathcal{J}^5_{13}$ is given in Table 6.

Finally, assume that one of the spaces in the decomposition $R = J_{12} \oplus J_{13} \oplus J_{23}$ (say J_{23}) has dimension 2. Then $R = J_{23}$, $J_{1j} = 0$ for $j = 2, 3$, thus $J_{11} = \mathbb{C}\, e_1$. As above, one proves that J is isomorphic to the direct product $J_{11} \times J'$. Since J has rank 3, J' has rank 2 and $R = J_{23} = \mathrm{Rad}(J')$ is 2-dimensional. Thus J is isomorphic to the Jordan algebra $\mathcal{J}^5_{14} = \mathbb{C} \times \mathcal{J}^4_{Q,1}$. The associated Jordan adjoint (in standard coordinates) is given in Table 6.

Let us now consider the case $\dim(R) = 1$. In this case, J_{ss} has rank 2 or 3. Moreover, since it admits a non-trivial cubic norm, it cannot be isomorphic to $\mathcal{J}^4_{Q,3}$ (which does not admit any) hence it is necessarily of rank 3. It follows that one can assume that J_{ss} is the direct product $\mathbb{C} \times (\mathbb{C} \oplus W)$ where $\mathbb{C} \oplus W = \mathcal{J}^3_{Q,2}$. Let a be a non-trivial element of R. Since $a^3 = 0$ and $a^2 \in R^2 \subset R = \mathbb{C}\, a$, it follows that $a^2 = 0$. Set $W = \mathbb{C}^2$ and let (x, y, z, t, u) be the usual system of coordinates on $J = (\mathbb{C} \times (\mathbb{C} \oplus W)) \oplus \mathbb{C}\, a$ (these coordinates are such that $(0, y, z, t, 0) \cdot (0, \tilde{y}, \tilde{z}, \tilde{t}, 0) = (0, y\tilde{y} + z\tilde{z} + t\tilde{t}, y\tilde{z} + z\tilde{y}, y\tilde{t} + t\tilde{y}, 0)$ for every $y, \tilde{y}, z, \tilde{z}, t, \tilde{t} \in \mathbb{C}$). Then $e_1 = (1, 0, 0, 0, 0)$, $e_2 = \frac{1}{2}(0, 1, 1, 0, 0)$ and $e_3 = \frac{1}{2}(0, 1, -1, 0, 0)$ are pairwise orthogonal primitive idempotents of J such that $e = e_1 + e_2 + e_3$.

The direct sum $J' = \oplus_{i,j=2}^{3} J_{ij}$ is a subalgebra of J. If $a \in J'$, one proves that J is isomorphic to the direct product of J_{11} with J'. If $\dim J_{11} > 1$, we would have $3 = \mathrm{rk}(J) = \mathrm{rk}(J_{11}) + \mathrm{rk}(J') \geqslant 2 + 2 = 4$, a contradiction. Hence $J_{11} \simeq \mathbb{C}$ and J' is a rank 2 Jordan of dimension 4 with a 1-dimensional radical. Hence J is isomorphic to the Jordan algebra $\mathcal{J}^5_{15} = \mathbb{C} \times \mathcal{J}^4_{Q,2}$ and there is then no difficulty to get an explicit expression for the adjoint in some coordinates (see Table 6 below).

Suppose now that $a \in J_{11} \oplus J_{12} \oplus J_{13}$. The case $a \in J_{11}$ does not occur (it would imply that J is isomorphic to $J_{11} \times J'$ hence we would have

$\mathrm{rk}(J) = \mathrm{rk}(J_{11}) + \mathrm{rk}(J') \geqslant 2 + 2$, a contradiction), hence $a \in J_{12} \oplus J_{13}$. By exchanging e_2 and e_3, if necessary, one can assume that $a \in J_{12}$. Set $\tilde{e} = (0, 0, 0, 1, 0) \in J_{\mathrm{ss}} = (\mathbb{C} \times (\mathbb{C} \oplus W))$. One verifies that $\tilde{e} \in J_{23}$ and that $\tilde{e}^2 = e_2 + e_3$. Moreover, since $\tilde{e} \cdot a \in J_{23} \cdot J_{12} \subset J_{13}$ and since $J_{13} = 0$ (for dimensional reasons), it follows that $\tilde{e} \cdot a = 0$. From this one deduces that the multiplication table of J in the basis $(e_1, e_2, e_3, \tilde{e}, a)$ is the following:

	e_1	e_2	e_3	$\tilde{e}$	a
e_1	e_1				$\frac{1}{2}a$
e_2		e_2		$\frac{1}{2}\tilde{e}$	$\frac{1}{2}a$
e_3			e_3	$\frac{1}{2}\tilde{e}$	
$\tilde{e}$		$\frac{1}{2}\tilde{e}$	$\frac{1}{2}\tilde{e}$	$e_2 + e_3$	
a	$\frac{1}{2}a$	$\frac{1}{2}a$			

By direct computations, one verifies that the multiplicative product defined by this table does not satisfy the Jordan identity. Thus the case when $a \in J_{11} \oplus J_{12} \oplus J_{13}$ does not occur.

Finally, when $\dim R = 0$, J is semi-simple hence is isomorphic to the direct product $\mathcal{J}_{16}^5 = \mathbb{C} \times \mathcal{J}_{Q,3}^4$.

We have thus finally obtained the classification of rank 3 Jordan algebras of dimension 5:

THEOREM 3.4. — *A rank 3 Jordan algebra of dimension 5 is isomorphic to exactly one of the sixteen algebras $\mathcal{J}_k^5$'s described above.*

3.4.3. Classification of quadro-quadric Cremona transformations of $\mathbb{P}^4$

Using the *JC*-equivalence, one deduces from the previous result the complete classification of quadro-quadric Cremona transformations of $\mathbb{P}^4$. Furthermore, one verifies that the Hilbert polynomials of the base locus schemes $\mathcal{B}_I, \mathcal{B}_{II}$ and $\mathcal{B}_{III}$ are respectively

$$h_I = P_0 - P_1 + 2P_2\,, \quad h_{II} = -2P_0 + 2P_1 + P_2 \quad \text{and} \quad h_{III} = -5P_0 + 5P_1.$$

In particular, these are distinct, thus the type of a quadro-quadric Cremona transformation of $\mathbb{P}^4$ given explicitly can be determined easily by computing the Hilbert polynomial of its base locus scheme (using the software *MacCaulay2* [15] for instance). Similarly, there is no difficulty in determining the multidegree of such a Cremona transformation f: the intersection of two distinct quadrics in $f^{-1}|\mathcal{O}_{\mathbb{P}^4}(1)|$ is the scheme theoretic

union of $\mathcal{B}_f$ with another scheme whose degree is the integer k such that $\mathrm{mdeg}(f) = (2, k, 2)$.

THEOREM 3.5. — *A quadro-quadric Cremona transformation of $\mathbb{P}^4$ is linearly equivalent to the projectivization of one of the sixteen (pairwise non linearly equivalent) Jordan involutions in the following table:*

Algebra	**dim R**	**Jordan involution $(x, y, z, t, u)^{\#}$**	**Type**	**Multidegree**
$\mathcal{J}_1^5$	4	$(x^2\,, -xy\,, -xz\,, y^2 - xt\,, 2yz - xu)$	II	$(2, 3, 2)$
$\mathcal{J}_2^5$	4	$(x^2\,, -xy\,, -xz\,, y^2 + z^2 - xt\,, 2yz - xu)$	III	$(2, 4, 2)$
$\mathcal{J}_3^5$	4	$(x^2\,, -xy\,, -xz\,, -xt\,, y^2 - xu)$	I	$(2, 2, 2)$
$\mathcal{J}_4^5$	4	$(x^2\,, -xy\,, -xz\,, -xt\,, y^2 + z^2 - xu)$	I	$(2, 2, 2)$
$\mathcal{J}_5^5$	4	$(x^2\,, -xy\,, -xz\,, -xt\,, y^2 + z^2 + t^2 - xu)$	I	$(2, 2, 2)$
$\mathcal{J}_6^5$	3	$(xy\,, x^2\,, -yz\,, -xt\,, -xu)$	I	$(2, 2, 2)$
$\mathcal{J}_7^5$	3	$(xy\,, x^2\,, t^2 - yz\,, -xt\,, -xu)$	I	$(2, 2, 2)$
$\mathcal{J}_8^5$	3	$(xy\,, x^2\,, t^2 + u^2 - yz\,, -xt\,, -xu)$	I	$(2, 2, 2)$
$\mathcal{J}_9^5$	3	$(xy\,, x^2\,, -yz\,, -xt\,, 2zt - xu)$	II	$(2, 3, 2)$
$\mathcal{J}_{10}^5$	3	$(xy\,, x^2\,, -yz\,, -yt\,, -xu)$	II	$(2, 3, 2)$
$\mathcal{J}_{11}^5$	3	$(xy\,, x^2\,, u^2 - yz\,, -yt\,, -xu)$	III	$(2, 4, 2)$
$\mathcal{J}_{12}^5$	3	$(xy\,, x^2\,, -yz\,, -yt\,, -yu)$	I	$(2, 2, 2)$
$\mathcal{J}_{13}^5$	2	$(yz\,, xz\,, xy\,, -zt\,, -yu)$	II	$(2, 3, 2)$
$\mathcal{J}_{14}^5$	2	$(y^2 + z^2\,, xy\,, -xz\,, -xt\,, -xu)$	I	$(2, 2, 2)$
$\mathcal{J}_{15}^5$	1	$(y^2 + z^2 + t^2\,, xy\,, -xz\,, -xt\,, -xu)$	I	$(2, 2, 2)$
$\mathcal{J}_{16}^5$	0	$(y^2 + z^2 + t^2 + u^2\,, xy\,, -xz\,, -xt\,, -xu)$	I	$(2, 2, 2)$

Table 6. Classification of quadro-quadric Cremona transformations of $\mathbb{P}^4$.

The classification above is completely explicit but does not say much about the geometry of the corresponding Cremona transformations. In the next subsections, for each Jordan involution $f_{\mathcal{J}_k^5}$ in this table, we describe as geometrically (and so little schematically) as possible its base locus scheme $\mathcal{B}(\mathcal{J}_k^5)$ as well as the linear system of quadric hypersurfaces

$$\big|\mathcal{I}_{\mathcal{B}(\mathcal{J}_k^5)}(2)\big| = f_{\mathcal{J}_k^5}^{-1}\big|\mathcal{O}_{\mathbb{P}^4}(1)\big|.$$

3.4.4. Tables of quadro-quadric Cremona transformations of $\mathbb{P}^4$

We will deal with the three types I, II and III separately. The results are collected in some tables below according to their type in a similar way

to the classification tables of Cremona transformations of type $(2, d)$ of $\mathbb{P}^3$ obtained in [28]. When possible, we also offer graphic representations of the associated base locus schemes. These drawings could be helpful to give a better geometrical understanding of various cases, especially the most degenerated ones.

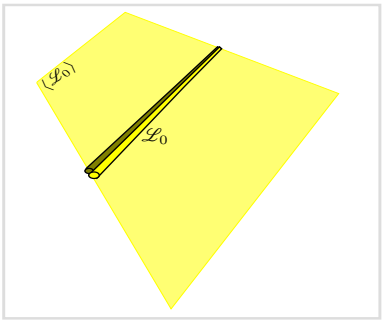
Some Cremona maps of type II or III are rather complicated since their base locus schemes have a non-reduced irreducible component whose schematic structure is more subtle than for type I. Before presenting the aforementioned tables of Cremona transformations, we introduce some terminology in order to deal with some non-reduced schemes of dimension 1.

– *Double structures on a line in* $\mathbb{P}^3$. Let $\mathscr{L}$ be a double structure supported on a line $\ell = \mathscr{L}_{\mathrm{red}} \subset \mathbb{P}^3$ (see [29, Section 1]): the associated ideal sheaf $\mathcal{I}_{\mathscr{L}}$ verifies $(\mathcal{I}_\ell)^2 \subset \mathcal{I}_{\mathscr{L}} \subset \mathcal{I}_\ell$ and is such that the sheaf $\mathcal{J}_{\mathscr{L}} = \mathcal{I}_\ell/\mathcal{I}_{\mathscr{L}}$ is invertible. Thus there is a surjection $\mathcal{I}_\ell/(\mathcal{I}_\ell)^2 \to \mathcal{J}_{\mathscr{L}}$ to which corresponds a unique section $\sigma_{\mathscr{L}} \in H^0(\ell, N_{\ell/\mathbb{P}^3})$. It defines a line bundle on $\ell \simeq \mathbb{P}^1$ whose degree $\delta(\mathscr{L}) \geqslant 0$ is projectively attached to $\mathscr{L} \subset \mathbb{P}^3$. In fact, it is the unique invariant of such a scheme since setting $\delta = \delta(\mathscr{L})$ for simplicity, one knows (cf. [11, page 136] for instance) that $\mathscr{L}$ is projectively isomorphic to

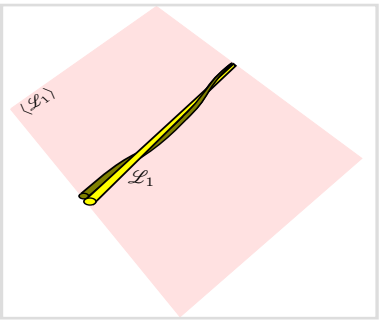
$$\mathscr{L}_\delta = \mathrm{Proj}\left(\frac{\mathbb{C}[a, b, d, c]}{(a^2, ab, b^2, ad^\delta - bc^\delta)}\right) \subset \mathbb{P}^3.$$

One can describe geometrically $\mathscr{L}_\delta$ as the union of the rational family of punctual schemes $\{\tau_x\}_{x\in\ell}$, all isomorphic to $\tau \subset \mathbb{P}^3$ and such that the projectivized tangent space $T_x\tau_x$ is *normal* to ℓ for every x. As mentioned in [11], $\mathscr{L}$ can be thought as ℓ 'plus a projective line $\mathcal{L}$ infinitely close to it', the integer δ specifying how fast $\mathcal{L}$ 'twists along ℓ' in $\mathbb{P}^3$. When $\delta = 0$, there is no twisting at all so that, using the terminology introduced in Section 1, $\mathscr{L}_0$ is nothing but a 2-multiple line in a 2-plane, *i.e.* $\mathscr{L}_0 \simeq \mathrm{Proj}(\mathbb{C}[a, b, c]/(a^2)) \subset \mathbb{P}^2$. This contrasts with the general case since $\langle \mathscr{L}_\delta \rangle = \mathbb{P}^3$ for every positive integer δ.

In what follows, we will consider only the schemes $\mathscr{L}_\delta$ for $\delta = 0, 1$, linearly embedded in $\mathbb{P}^4$. According to the above discussion, it is reasonable to represent these two schemes as: a line plus another one glued along the first, in a rectilinear way for $\delta = 0$; twisted one time when $\delta = 1$:



Pictural representation of $\mathscr{L}_0 \subset \mathbb{P}^4$
(the 2-plane $\langle \mathscr{L}_0 \rangle$
is pictured in yellow)



Pictural representation of $\mathscr{L}_1 \subset \mathbb{P}^4$
(the hyperplane $\langle \mathscr{L}_1 \rangle$
is pictured in pink)

– *The non-reduced degenerate rational quartic curve* $\mathcal{L}^4 \subset \mathbb{P}^4$. Let

$$\mathcal{L}^4 = \mathrm{Proj}\Big(\mathbb{C}[x,y,z,t,u]/(x^2\,,\,xy\,,\,xz\,,\,y^2-xt\,,\,2yz-xu,z^2)\Big) \subset \mathbb{P}^4.$$

This subscheme of $\mathbb{P}^4$ is irreducible and has dimension 1. It is an arithmetically Cohen-Macaulay (briefly ACM) non reduced curve of degree 4 having arithmetic genus 0 and supported on the line $\ell = \mathscr{L}^4_{\mathrm{red}} = V(x,y,z) \subset \mathbb{P}^4$. For any point $p \in \ell$, the projectivized tangent space $T_p\mathcal{L}^4$ is the coordinate hyperplane $H = V(x)$. The (schematic) intersection $\mathcal{L}^4 \cap H$ is the scheme defined by the homogeneous ideal (x, y^2, yz, z^2): it is a double-line in H. The residual intersection of H with $\mathcal{L}^4$ is the reduced line ℓ.

A natural question is to wonder if $\mathcal{L}^4$ is a degeneration of a rational normal quartic curve $v_4(\mathbb{P}^1) \subset \mathbb{P}^4$, which has the same Hilbert polynomial. This should follow from [24, Proposition 2.2] but the algorithm presented there for producing a smoothing of the scheme $\mathcal{L}^4$ does not work because in this case a non flat family appears[(8)]. Thus as far as we know, it is still unknown if $\mathcal{L}^4$ belongs to the irreducible component of the Hilbert scheme containing $v_4(\mathbb{P}^1)$ or not. This shows how subtle the analysis of the most degenerated examples can be.

[(8)] Actually, the deformation $\mathcal{C}$ given in part 2) of the proof of [24, Proposition 2.2] is even not equidimensional in general.

In the next pages, the reader will find the aforementioned tables of quadro-quadric Cremona transformations of $\mathbb{P}^4$ as well as the corresponding graphic representations of the associated base locus schemes.

We begin by considering the quadro-quadric Cremona transformations of type I. The descriptions of the linear systems in TABLE 7 are complete and very similar to the ones in 'Tableau 1' of [28][(9)].

Then we consider the case of quadro-quadric Cremona transformations of type II. All the associated linear systems of quadrics are described in TABLE 8, with the exception of $|\mathcal{I}_{\mathcal{B}(\mathcal{J}_1^5)}|$ whose description is too long to be put in the table. In order to describe it geometrically, let us introduce the following terminology: if C is a curve included in the smooth locus of a surface $S \subset \mathbb{P}^4$, we will say that a hypersurface $\mathcal{H} \subset \mathbb{P}^4$ *osculates normally* S *along* C if any generic subscheme $\eta \subset S$ normal to C is also a subscheme of $\mathcal{H}$. More concretely, this means that for $c \in C$ generic and any regular germ of curve $\mu : (\mathbb{C}, 0) \to (S, c)$ whose image is transverse to C, the composition $h \circ \mu$ has valuation at least 2 at 0 if h is a generator of $\mathcal{O}_{\mathcal{H},c}$.

Now let ℓ be a fixed line of the ruling of a rational normal scroll $S = S_{1,2}$ in $\mathbb{P}^4$ and denote by π the 2-plane spanned by ℓ and the directrix line of S. Then up to projective equivalence,

the homaloidal linear system $|\mathcal{I}_{\mathcal{B}(\mathcal{J}_1^5)}|$ is formed by the quadric hypersurfaces in $\mathbb{P}^4$ containing the 2-plane π and osculating normally the rational normal scroll S along ℓ. (3.5)

This claim can be verified by easy direct computations[(10)].

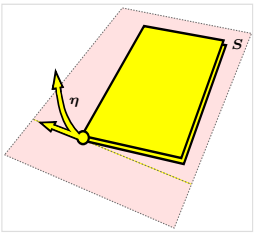
The base locus schemes of type II are pictured in FIGURE 6, at the exception of $\mathcal{B}(\mathcal{J}_1^5)$: we have not been able to figure a way to represent this scheme.

(9) Note however that the description of the linear system of type $\tan^{[3]}(//)$ is not correct in [28].

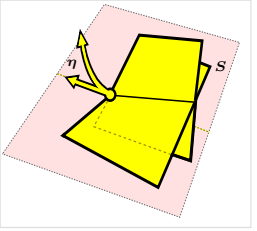
(10) To recover exactly the linear system $\langle x^2, xy, xz, y^2 - xt, 2yz - xu, z^2\rangle$ appearing in TABLE 8, one has to take $S = V(y^2 - xt, 2zt - yu, 2yz - xu)$ and $\ell = V(x, y, z)$.

Algebra $\mathcal{J}$	**Jordan adjoint $(x,y,z,t,u)^{\#}$**	**Primary decomposition of $\mathcal{I}_{\mathcal{B}(\mathcal{J})}$**	**Geometrical description of $\mathcal{B}(\mathcal{J})$**	**Linear system $\lvert\mathcal{I}_{\mathcal{B}(\mathcal{J})}(2)\rvert$**
$\mathcal{J}_3^5$	$(x^2\,,\,-xy\,,\,-xz\,,\,-xt\,,\,y^2-xu)$	$(x,y^2)\cap(t,z,xy,x^2,y^2-xu)$	A rank 1 quadric surface S plus a scheme η supported on S_{red} such that $\langle\eta\rangle\cap\langle S\rangle$ is the line tangent to η that is moreover transverse to S_{red}	Rank 3 quadric hypercones containing S_{red} with vertex line included in S_{red} and osculating along η
$\mathcal{J}_4^5$	$(x^2\,,\,-xy\,,\,-xz\,,\,-xt\,,\,y^2+z^2-xu)$	$(x,y-i\,z)\cap(x,y+i\,z)$ $\cap(t,z,xy,x^2,y^2-xu)$	A rank 2 quadric surface S plus a scheme η supported on the line S_{sing} such that $\langle\eta\rangle\cap\langle S\rangle$ is the line tangent to η that is not contained in S	Hyperquadrics containing S and osculating along η
$\mathcal{J}_5^5$	$(x^2\,,\,-xy\,,\,-xz\,,\,-xt\,,\,y^2+z^2+t^2-xu)$	$(x,y^2+z^2+t^2)\cap$ (t,z,xy,x^2,y^2-xu)	A rank 3 quadric surface S plus a scheme η supported at the vertex of S such that $\langle\eta\rangle\cap\langle S\rangle$ is the line tangent to η that is not included in S	Hyperquadrics containing S and osculating along η
$\mathcal{J}_6^5$	$(xy\,,\,x^2\,,\,-yz\,,\,-xt\,,\,-xu)$	$(x,y)\cap(x,z)\cap(y,t,u,x^2)$	A rank 2 quadric surface S plus a scheme τ supported at a point $p\in S_{\text{reg}}$ and spanning a line transverse to $\langle S\rangle$	Hyperquadrics containing S and tangent to $\langle T_pS,\tau\rangle$ at p
$\mathcal{J}_7^5$	$(xy\,,\,x^2\,,\,t^2-yz\,,\,-xt\,,\,-xu)$	$(x,t^2-yz)\cap(x^2,y,u,xt,t^2)$	A rank 3 quadric surface S plus a scheme ξ supported at a point $p\in S_{\text{reg}}$ such that $\langle\xi\rangle$ cuts $\langle S\rangle$ along a line tangent to S at p but not included in S	Hyperquadrics containing S and tangent to $\langle T_pS,\xi\rangle$ at p
$\mathcal{J}_8^5$	$(xy\,,\,x^2\,,\,t^2+u^2-yz\,,\,-xt\,,\,-xu)$	$(x,t^2+u^2-yz)\cap(x^2,y,u,xt,t^2)$	A smooth quadric surface S plus a scheme ξ supported at a point $p\in S$ such that $\langle\xi\rangle$ intersects $\langle S\rangle$ along a line tangent to S at p but not included in S	Hyperquadrics containing S and tangent to $\langle T_pS,\xi\rangle$ at p
$\mathcal{J}_{12}^5$	$(xy\,,\,x^2\,,\,-yz\,,\,-yt\,,\,-yu)$	$(x^2,y)\cap(x,z,t,u)$	A double plane S in a $\mathbb{P}^3$ plus a point p outside $\langle S\rangle$	Hyperquadrics tangent to $\langle S\rangle$ along S_{red} and passing through p
$\mathcal{J}_{14}^5$	$(y^2+z^2\,,\,xy\,,\,-xz\,,\,-xt\,,\,-xu)$	$(x,y^2+z^2)\cap(y,z,t,u)$	A rank 2 quadric surface S plus a point p outside $\langle S\rangle$	Hyperquadrics in $\mathbb{P}^4$ containing $S\cup\{p\}$
$\mathcal{J}_{15}^5$	$(y^2+z^2+t^2\,,\,xy\,,\,-xz\,,\,-xt\,,\,-xu)$	$(x,y^2+z^2+t^2)\cap(y,z,t,u)$	A quadric cone S in $\mathbb{P}^3$ plus a point p outside $\langle S\rangle$	Hyperquadrics in $\mathbb{P}^4$ containing $S\cup\{p\}$
$\mathcal{J}_{16}^5$	$(y^2+z^2+t^2+u^2\,,\,xy\,,\,-xz\,,\,-xt\,,\,-xu)$	$(x,y^2+z^2+t^2+u^2)\cap(y,z,t,u)$	A smooth quadric surface S plus a point p outside $\langle S\rangle$	Hyperquadrics in $\mathbb{P}^4$ containing $S\cup\{p\}$

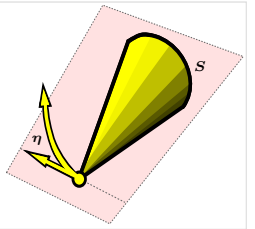
Table 7. Classification of elementary quadro-quadric Cremona transformations of $\mathbb{P}^4$ (Type I).



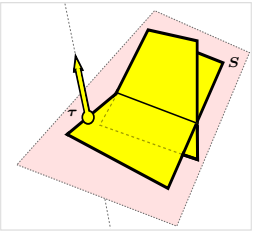
Base locus scheme $\mathcal{B}(\mathcal{J}_3^5)$



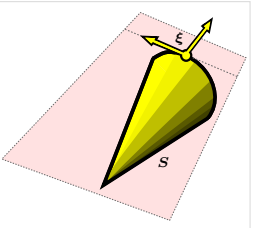
Base locus scheme $\mathcal{B}(\mathcal{J}_4^5)$



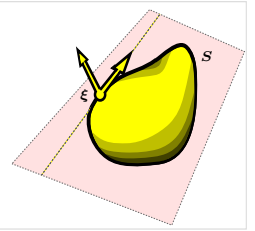
Base locus scheme $\mathcal{B}(\mathcal{J}_5^5)$



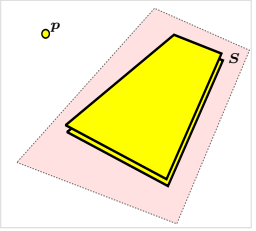
Base locus scheme $\mathcal{B}(\mathcal{J}_6^5)$



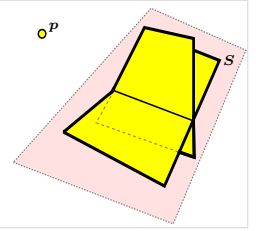
Base locus scheme $\mathcal{B}(\mathcal{J}_7^5)$



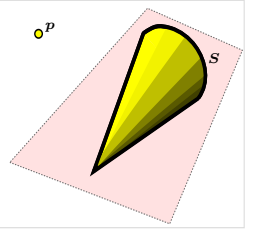
Base locus scheme $\mathcal{B}(\mathcal{J}_8^5)$



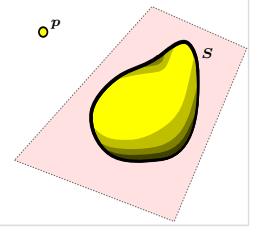
Base locus scheme $\mathcal{B}(\mathcal{J}_{12}^5)$



Base locus scheme $\mathcal{B}(\mathcal{J}_{14}^5)$



Base locus scheme $\mathcal{B}(\mathcal{J}_{15}^5)$

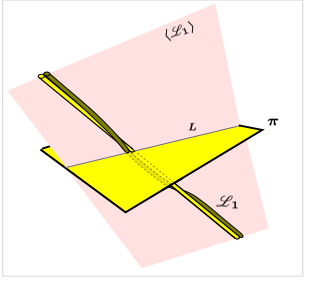


Base locus scheme $\mathcal{B}(\mathcal{J}_{16}^5)$

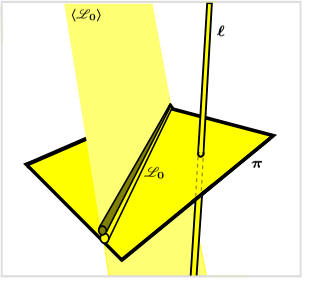
Figure 5. Graphic representations of the base locus schemes of elementary quadro-quadric Cremona transformations of $\mathbb{P}^4$ (these base loci are pictured in yellow and the hyperplanes spanned by the quadric surface components are pictured in pale pink).

Algebra $\mathcal{J}$	Quadro-quadric Cremona involution	Primary decomposition of $\mathcal{I}_{\mathcal{B}(\mathcal{J})}$	Geometrical description of $\mathcal{B}(\mathcal{J})$	Linear system $\lvert\mathcal{I}_{\mathcal{B}(\mathcal{J})}(2)\rvert$
$\mathcal{J}_1^5$	$(x^2\,,\,-xy\,,\,-xz\,,\,y^2-xt\,,\,2yz-xu)$	$(x,y)\cap(x^2\,,\,xy\,,\,xz\,,\,y^2-xt\,,\,2yz-xu,z^2)$	A 2-plane π plus a non-reduced degenerate rational quartic $\mathcal{L}^4$ with $\mathcal{L}^4_{red}$ included in π	Cf. (3.5)
$\mathcal{J}_9^5$	$(xy\,,\,x^2\,,\,-yz\,,\,-xt\,,\,2zt-xu)$	$(x,z)\cap(y,x^2,xt,t^2,2zt-xu)$	A 2-plane π plus a double structure $\mathscr{L}_1$ on a line such that the hyperplane $\langle\mathscr{L}_1\rangle$ cuts π along a line L that meets $(\mathscr{L}_1)_{\text{red}}$	Hyperquadrics containing π and tangent along $(\mathscr{L}_1)_{\text{red}}$ to a fixed smooth quadric surface in $\langle\mathscr{L}_1\rangle$ containing $L\cup(\mathscr{L}_1)_{\text{red}}$
$\mathcal{J}_{10}^5$	$(xy\,,\,x^2\,,\,-yz\,,\,-yt\,,\,-xu)$	$(x,y)\cap(x,z,t)\cap(u,y,x^2)$	A 2-plane π plus a line ℓ intersecting π plus a double line $\mathscr{L}_0$ such that the 2-plane $\langle\mathscr{L}_0\rangle$ does not meet ℓ and intersects π along $(\mathscr{L}_0)_{\text{red}}$	Hyperquadrics containing $\pi\cup\ell$ and tangent to $\langle\mathscr{L}_0\rangle$ along $(\mathscr{L}_0)_{\text{red}}$
$\mathcal{J}_{13}$	$(yz\,,\,xz\,,\,xy\,,\,-zt\,,\,-yu)$	$(u,z,x)\cap(t,y,x)\cap(z,y)$	A 2-plane π plus two skew lines ℓ,ℓ' intersecting π in two distinct points	Hyperquadrics in $\mathbb{P}^4$ containing $\pi\cup\ell\cup\ell'$

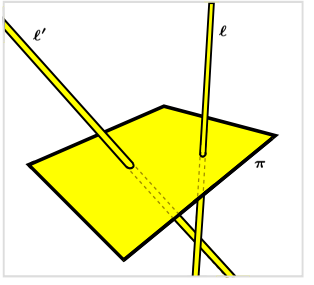
Table 8. Classification of quadro-quadric Cremona transformations of $\mathbb{P}^4$ of type II.



Base locus scheme $\mathcal{B}(\mathcal{J}_9^5)$



Base locus scheme $\mathcal{B}(\mathcal{J}_{10}^5)$



Base locus scheme $\mathcal{B}(\mathcal{J}_{13}^5)$

Figure 6. Pictural representations (in yellow) of the base locus schemes of quadro-quadric Cremona transformations of $\mathbb{P}^4$ of type II.

The classification shows that up to linear equivalence, there are only two quadro-quadric Cremona transformations of $\mathbb{P}^4$ of type III. The generic one is well-understood: the associated algebra is $\mathcal{J}^5_{11}$, the adjoint reads $(x,y,z,t,u)^\# = (xy, x^2, u^2 - yz, -yt, -xu)$ in a suitable system of coordinates, the primary decomposition of the associated ideal is $\mathcal{I}_{\mathcal{B}(\mathcal{J}^5_{11})} = (x,t,u^2-yz)\cap(y,x^2,xu,u^2)$, the base locus scheme $\mathcal{B}(\mathcal{J}^5_{11})$ is the scheme $\mathcal{B}_{III}$ of Theorem 3.3 and the associated linear system is formed by hyperquadrics containing C and tangent to the line $\mathcal{L}_{\mathrm{red}}$ at the intersection locus $C\cap\langle\mathcal{L}\rangle$, which is just one point.

In contrast we have almost nothing to say on the quadro-quadric Cremona transformations of $\mathbb{P}^4$ associated to the algebra $\mathcal{J}^5_2$: its base locus scheme $\mathcal{B}(\mathcal{J}^5_2)$ is a degree 5 multiplicity 2 scheme of arithmetic genus 1 supported on the line $V(x,y,z)$. This case appears to be the most degenerate and the most mysterious one.

Note however than if for $\lambda\in\mathbb{C}$, one defines

$$f_\lambda : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4\,, [x:y:z:y:u] \mapsto [x^2 : -xy : -xz : y^2+\lambda z^2-xt : 2yz-xu]\,,$$

one obtains an algebraic 1-dimensional family of quadro-quadric Cremona involutions of $\mathbb{P}^4$ such that $f_0 = f_{\mathcal{J}^5_1}$ and with f_λ linearly equivalent to $f_1 = f_{\mathcal{J}^5_2}$ for every $\lambda\neq 0$. Thus $f_{\mathcal{J}^5_1}$ naturally appears as a degeneration of $f_{\mathcal{J}^5_2}$ in contrast with what is going on at the schematic level. Indeed, the associated family of base locus schemes $\{\mathcal{B}_{f_\lambda}\}_{\lambda\in\mathbb{C}}$ is not flat (it is neither equidimensional since $\dim\mathcal{B}_{f_0}=2$ whereas $\dim\mathcal{B}_{f_\lambda}=1$ for every $\lambda\neq 0$).

The preceding remarks show that the study of the 'moduli space of quadro-quadric Cremona transformations of $\mathbb{P}^n$' is certainly quite difficult from a scheme theoretic perspective as soon as n increases.

Let us conclude this section with a general remark suggested by the referee: looking at Table 3, Table 6 and Table 10 below, one realizes that the Cremona transformations corresponding to the maximal dimension of R are exactly those coming from a quadratic polynomial automorphism of the affine space (given for all cases as the complement of the hyperplane $x=0$, that is by the chart $x=1$). This is not a coincidence and there is a conceptual reason why this holds true (in any dimension). Indeed, let $f:\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the projectivization of the adjoint map of a Jordan algebra J of rank 3. If $\mathcal{N}$ stands for the support of the (possibly non-irreducible or non-reduced) cubic hypersurface cut out by the cubic norm of J, then, by restriction, f induces an isomorphism $\mathbb{P}^n\setminus\mathcal{N}\to\mathbb{P}^n\setminus\mathcal{N}$. And $\mathbb{P}^n\setminus\mathcal{N}$ is an affine space if and only if $\mathcal{N}$ is a hyperplane which is equivalent to the fact that the radical of J is of codimension 1.

4. Quadro-quadric Cremona transformations of $\mathbb{P}^5$

In [5], Bruno and Verra apply Semple's approach (see Section 3.4.1) to the case of $\mathbb{P}^5$. Using the description of quadric Cremona transformations of $\mathbb{P}^4$ given by Semple in [35], they prove the following result:

THEOREM 4.1. — *The base locus scheme of a quadro-quadric Cremona transformation of $\mathbb{P}^5$ belongs to the irreducible components of* $\mathrm{Hilb}(\mathbb{P}^5)$ *containing one of the subschemes $\mathcal{B}_I, \dots, \mathcal{B}_{IV}$ of $\mathbb{P}^5$ where:*

(I) *$\mathcal{B}_I$ is the disjoint union of a smooth quadric threefold Q with a point p lying outside the hyperplane $\langle Q \rangle$;*

(II) *$\mathcal{B}_{II}$ is the union $\Pi \cup \pi \cup \ell$ of a 3-plane Π with a 2-plane π and a line ℓ with relative positions as follows: π intersects Π along a line, ℓ is disjoint from π and meets Π at one point;*

(III) *$\mathcal{B}_{III}$ is the schematic union of a double plane $\mathcal{P}$ in a hyperplane H with a smooth conic C that is tangent to H at $C \cap \mathcal{P}_{\mathrm{red}}$ that is one point;*

(IV) *$\mathcal{B}_{IV}$ is the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$.*

The preceding result implies in particular that there are exactly four types for elements of $\mathbf{Bir}_{22}(\mathbb{P}^5)$, that will be denoted by I, II, III and IV. As in dimension four, to these types correspond four generic quadro-quadric Cremona transformations of $\mathbb{P}^5$. Explicit involutive normal forms for these as well as the corresponding multidegrees are given in the following table:

Type T	**Base locus $\mathcal{B}_T$**	**Generic Cremona transformation f_T**	**Multidegree**
I	$Q \sqcup \{p\}$	$(y^2+z^2+t^2+u^2+v^2, xy, -xz, -xt, -xu, -xv)$	$(2,2,2,2)$
II	$\Pi \cup \pi \cup \ell$	$(zy, xz, xy, -tz, -uz, -vy)$	$(2,3,3,2)$
III	$\mathcal{P} \cup C$	$(xy, x^2, -yz+v^2, -yt, -yu, -xv)$	$(2,4,4,2)$
IV	$v_2(\mathbb{P}^2)$	$(yz-v^2, xz-u^2, xy-t^2, uv-zt, tv-uy, tu-xv)$	$(2,4,4,2)$

Table 9. The four generic quadro-quadric Cremona transformations of $\mathbb{P}^5$.

A complete classification (up to isomorphisms) of rank 3 Jordan algebras of dimension 6 is obtained in [30] (following the more general classification done by Wesseler in [40]). One deduces from it the complete classification of quadro-quadric Cremona transformations of $\mathbb{P}^5$.

THEOREM 4.2. — *A quadro-quadric Cremona transformation of $\mathbb{P}^5$ is linearly equivalent to one of the twenty nine Jordan involutions listed in* TABLE *10 below.*

A careful and systematic study of the Cremona maps in TABLE 10 will be considered eventually elsewhere.

Algebra	Jordan adjoint $(x,y,z,t,u,v)^{\#}$	Type	Multidegree	$\dim(R)$
$\mathcal{J}_{\mathrm{s}}$	$\big(yz-v^2, xz-u^2, xy-t^2, uv-zt, tv-uy, tu-xv\big)$	IV	$(2,4,4,2)$	0
$\mathcal{J}_{\mathrm{ss}}$	$\big(y^2+z^2+t^2+u^2+v^2, xy, -xz, -xt, -xu, -xv\big)$	I	$(2,2,2,2)$	0
$\mathbb{C}\times\mathcal{J}^5_{Q,4}$	$\big(y^2+z^2+t^2+u^2, xy, -xz, -xt, -xu, -xv\big)$	I	$(2,2,2,2)$	1
$\mathbb{C}\times\mathcal{J}^5_{Q,3}$	$\big(y^2+z^2+t^2, xy, -xz, -xt, -xu, -xv\big)$	I	$(2,2,2,2)$	2
$\mathcal{J}_{92}$	$\big(yz-t^2, xz, xy, -xt, tv-uz, tu-yv\big)$	IV	$(2,4,4,2)$	2
$\mathcal{J}_{100}$	$(yz, xz, xy, -xt, -xu, -xv)$	I	$(2,2,2,2)$	3
$\mathcal{J}_{101}$	$(yz, xz, xy, -zt, -zu, -yv)$	II	$(2,3,3,2)$	3
$\mathcal{J}^0_{102}$	$(yz, xz, xy, -zt, -yu, -xv)$	IV	$(2,4,4,2)$	3
$\mathcal{J}^1_{102}$	$(yz, xz, xy, -zt+2uv, -yu, -xv)$	IV	$(2,4,4,2)$	3
$\mathcal{J}^{000}_{113}$	$(xy, x^2, -yz, -ty, -yu, -xv)$	II	$(2,3,3,2)$	4
$\mathcal{J}^{100}_{100}$	$(xy, x^2, -yz+v^2, -yt, -yu, -xv)$	III	$(2,4,4,2)$	4
$\mathcal{J}^a_{122}$	$\big(xy, x^2, -yz, -ty, -xu, -xv\big)$	II	$(2,3,3,2)$	4
$\mathcal{J}^b_{122}$	$\big(xy, x^2, -yz+u^2, -yt, -xu, -xv\big)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^c_{122}$	$\big(xy, x^2, -yz+2uv, -yt, -xu, -xv\big)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^d_{122}$	$\big(xy, x^2, -yz+u^2, -yt+v^2, -xu, -xv\big)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^e_{122}$	$\big(xy, x^2, -yz+u^2, -yt+2uv, -xu, -xv\big)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^0_{124}$	$(xy, x^2, -yz, -yt, -xu, -xv+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^1_{124}$	$(xy, x^2, -yz+u^2, -yt, -xu, -xv+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^0_{126}$	$(xy, x^2, -yz, -yt, -xu, -xv+2zu+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^1_{126}$	$(xy, x^2, -yz+u^2, -yt-u^2, -xu, -xv+2zu+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^{(0)}_{140}$	$(xy, x^2, -yz, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(1)}_{140}$	$(xy, x^2, -yz+t^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(2)}_{140}$	$(xy, x^2, -yz+t^2+u^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(3)}_{140}$	$(xy, x^2, -yz++t^2+u^2+v^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}_{143}$	$(xy, x^2, -yz, -xt, -xu, -xv+2zt)$	II	$(2,3,3,2)$	4
$\mathcal{J}^a_{ES}$	$\big(x^2, -xy, -xz, -xt+2yu-zv, -xu+z^2, -xv+2yz\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^b_{ES}$	$\big(x^2, -xy, -xz, -xt+2yz, -xu+y^2, -xv+z^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(-1)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu-2yz+2zt, -xv-y^2+t^2\big)$	II	$(2,3,3,2)$	5
$\mathcal{J}^{c,(0)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+2zt, -xv+t^2\big)$	II	$(2,3,3,2)$	5
$\mathcal{J}^{c,(d)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+2zt, -xv+z^2+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(1)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+y^2+2zt, -xv+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(2)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+2zt, -xv+y^2+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(3)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+2yz+2zt, -xv+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(4)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+2zt, -xv+2yz+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(5)}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu+y^2+2zt, -xv+z^2+t^2\big)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{d,000}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu, -xv+y^2\big)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,100}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2\big)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,110}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2+t^2\big)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,111}_{ES}$	$\big(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2+t^2+u^2\big)$	I	$(2,2,2,2)$	5

Table 10. Involutive normal forms for quadro-quadric Cremona transformations of $\mathbb{P}^5$ ($\dim(R)$ stands for the dimension of the radical of the corresponding Jordan algebra, this one being labelled with the notation used in [30]*).*

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