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NON-ABELIAN CONGRUENCES BETWEEN L-VALUES OF ELLIPTIC CURVES

by Daniel DELBOURGO & Tom WARD (*)

ABSTRACT. — Let E be a semistable elliptic curve over $\mathbb{Q}$. We prove weak forms of Kato's K_1-congruences for the special values $L\Big(1, E/\mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{\Delta})\Big)$. More precisely, we show that they are true modulo p^{n+1}, rather than modulo p^{2n}. Whilst not quite enough to establish that there is a non-abelian L-function living in $K_1\Big(\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\mu_{\mathrm{p}^\infty}, \sqrt[p^\infty]{\Delta})/\mathbb{Q})]]\Big)$, they do provide strong evidence towards the existence of such an analytic object. For example, if $n = 1$ these verify the numerical congruences found by Tim and Vladimir Dokchitser.

RÉSUMÉ. — Soit E une courbe elliptique définie sur $\mathbb{Q}$. Nous démontrons des versions faibles des congruences K_1 de Kato, pour les valeurs spéciales $L\Big(1, E/\mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{\Delta})\Big)$. Plus précisément, nous vérifions que les congruences sont vraies modulo p^{n+1}, plutôt que modulo p^{2n}. Bien que ça ne suffise pas pour établir l'existence d'une fonction L p-adique qui vit dans $K_1\Big(\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\mu_{\mathrm{p}^\infty}, \sqrt[p^\infty]{\Delta})/\mathbb{Q})]]\Big)$, elles fournissent beaucoup d'indices de l'existence de cet objet analytique. Par exemple, si $n = 1$ les congruences trouvées numériquement par Tim et Vladimir Dokchitser sont vraies.

1. Introduction

In this paper, we study the behaviour of the Hasse-Weil L-functions of elliptic curves, over the so-called "False Tate Curve" extensions of $\mathbb{Q}$. These are non-abelian p-adic Lie extensions of dimension two, which may be constructed as follows.

Fix a prime number $p \neq 2$. Let $\Delta > 1$ denote a p-power free integer. We suppose that Δ is coprime to p, which ensures all the primes above Δ

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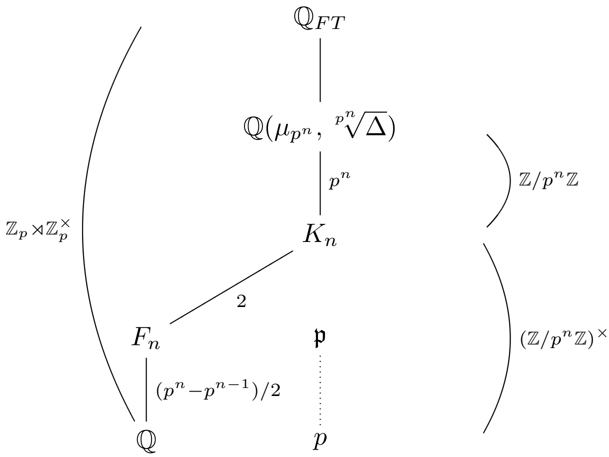
tamely ramify in the false Tate curve tower. For each integer $n \geqslant 0$, we set $K_n = \mathbb{Q}(\mu_{p^n})$ and write $F_n = \mathbb{Q}(\mu_{p^n})^+$ for the maximal real subfield. Then

$$\mathbb{Q}_{FT} = \bigcup_{n\geqslant 1} \mathbb{Q}\left(\mu_{p^n},\ \sqrt[p^n]{\Delta}\right).$$

Basic Galois theory informs us that

$$\mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q}) \cong \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \lhd \mathrm{GL}_2(\mathbb{Z}_p).$$

In other words, the Galois group is a semi-direct product of two p-adic Lie groups of dimension one. In terms of a field diagram,



In this situation the representation theory is very well understood. It is proved in [6] that $\mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q})$ has a unique self-dual representation of dimension $p^k - p^{k-1}$, which we denote by $\rho_{k,\mathbb{Q}}$ for each $k \geqslant 1$. This may be written

$$\rho_{k,\mathbb{Q}} = \mathrm{Ind}_{K_k}^{\mathbb{Q}}\, \chi_{\rho_k}$$

for a character χ_{ρ_k} of $\mathrm{Gal}(\mathbb{Q}_{FT}/K_k)$. Putting $\rho_{0,\mathbb{Q}} = \mathbf{1}$, every irreducible representation of $\mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q})$ has the form $\rho_{k,\mathbb{Q}} \otimes \psi$ for some $k \geqslant 0$, and some character

$$\psi : \mathrm{Gal}\left(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}\right) \to \mathbb{C}^\times.$$

For the rest of this section, we set $G = \mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q})$.

Remark. — Let E be an elliptic curve over $\mathbb{Q}$. As part of a more general "GL_2-Main Conjecture", Coates et al [3] predict the existence of a non-abelian p-adic L-function

$$\mathbf{L}_p^{\mathrm{anal}}(E/\mathbb{Q}_{FT}) \in K_1(\mathbb{Z}_p[[G]]_{\mathcal{S}^*})$$

whose evaluation at Artin representations $\rho : G \to \mathrm{GL}(V)$ essentially yield the ρ-twisted L-values $L(1, E, \rho)$. Here $\mathbb{Z}_p[[G]]_{\mathcal{S}^*}$ is the localisation of $\mathbb{Z}_p[[G]]$ at a certain Ore set $\mathcal{S}^* = \bigcup_{n \geqslant 0} p^{-n}\mathcal{S}$.

In his beautiful paper [9], Kato reduced the question of existence for $\mathbf{L}_p^{\mathrm{anal}}$ into a sequence of congruence relations, amongst the abelian p-adic L-functions interpolating E over the false Tate curve extension. More precisely, if $U^{(n)} = \ker(\mathbb{Z}_p^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times)$ then there exists an injection

$$\Theta_{G,\mathcal{S}^*} : K_1(\mathbb{Z}_p[[G]]_{\mathcal{S}^*}) \overset{\prod(\rho_n)_*}{\longrightarrow} \prod_{n \geqslant 0} \mathrm{Quot}(\mathbb{Z}_p[[U^{(n)}]])^\times.$$

Kato calculated that a sequence $(a_n)_{n \geqslant 0}$ lies in the image of $\Theta_{G,\mathcal{S}^*}$ if and only if

$$\prod_{1 \leqslant i \leqslant n} N_{i,n} \left(\frac{a_i}{N_{0,i}(a_0)} . \frac{\phi \circ N_{0,i-1}(a_0)}{\phi(a_{i-1})} \right)^{p^i} \equiv 1 \mod p^{2n} \quad \text{for all } n \in \mathbb{N}.$$

Here we should point out that $N_{i,j} : \mathbb{Z}_p[[U^{(i)}]]^\times \to \mathbb{Z}_p[[U^{(j)}]]^\times$ denotes the norm map, and $\phi : \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the ring homomorphism induced by the p-power map on $\mathbb{Z}_p^\times$.

Let E be an elliptic curve defined over the rationals, in particular, it is modular by the work of Wiles et al. Moreover, we shall assume that E is semistable (otherwise our distributions turn out to be identically zero). We write f_E for the newform of weight two and conductor N_E associated to E. In order to state our full results, we are forced to make three assumptions about the prime p, and the field $F_n = \mathbb{Q}(\mu_{p^n})^+$.

Hypothesis (Ord). — The elliptic curve E has good ordinary reduction at p, with Δ and N_E coprime integers.

Hypothesis (p-Irr). — The base change $\mathbf{f}_{/F_n}$ of f_E to the field F_n is not congruent modulo $\mathfrak{p}$ to any other Hilbert modular form of the same level.

Hypothesis (Per). — Conjecture 1.3 from [4] holds for the cusp form f_E and the field F_n. This implies the automorphic and motivic periods of E are the same, up to a p-adic unit.

The third hypothesis has been verified numerically by Doi-Hida-Ishii in many cases. It is intimately connected to the orders of congruence modules for the space of Hilbert automorphic forms over F_n (see [4] for the precise statement).

Our first result is an integral version of a theorem of Shai Haran from [8], concerning the existence of abelian L-functions which are attached to the motives $h^1(E) \otimes_{\mathbb{Z}} M(\rho_n)$. Recall that for an Artin representation ρ over F_n, one defines the ρ-twisted L-function $L(s,E,\rho)$ by the infinite product

$$L(s,E,\rho) = \prod_v \det\left(1 - N_{F_n/\mathbb{Q}}(v)^{-s}\Phi_v \Big| \left(H_l^1(E) \otimes_{\overline{\mathbb{Q}}_l} \rho\right)^{I_v}\right)^{-1}$$

where Φ_v is a geometric Frobenius element for v, and I_v is the inertia group. This Euler product converges to an analytic function on $\mathrm{Re}(s) > 3/2$. In general, the L-functions of E and its twists are conjectured to have an analytic continuation to all of $\mathbb{C}$.

Theorem 1.1. — *Let $\mathfrak{p}$ denote the unique prime of F_n above p. Assume that E satisfies Hypothesis (Ord), and that in addition both of (p-Irr) and (Per) hold for f_E over F_n. Then there exists a unique element $\mathbf{L}_p(E,\rho_n) \in \mathbb{Z}_p[[U^{(n)}]]$ such that*

$$\psi(\mathbf{L}_p(E,\rho_n)) = \frac{\epsilon_{F_n}(\rho_n \otimes \psi)_{\mathfrak{p}}}{\alpha_p^{f(\rho_n\otimes\psi,\mathfrak{p})}} \times \frac{P_{\mathfrak{p}}(\rho_n \otimes \psi, \alpha_p^{-[F_n:\mathbb{Q}]})}{P_{\mathfrak{p}}(\rho_n \otimes \psi^{-1}, \alpha_p'^{-[F_n:\mathbb{Q}]})} \times \frac{L_S(1,E,\rho_n \otimes \psi^{-1})}{(\Omega_E^+\Omega_E^-)^{\phi(p^n)/2}}$$

for all finite characters ψ of $U^{(n)}$.

Note. — Here $P_{\mathfrak{p}}(\rho_n \otimes \psi, X)$ denotes the characteristic polynomial of $\Phi_{\mathfrak{p}}$ on the inertia invariant subspace, and α_p denotes the p-adic unit root of

$$X^2 - a_p(E)X + p$$

with α_p' being the non-unit root. Further, $\epsilon_{F_n}(\rho_n \otimes \psi)_{\mathfrak{p}}$ denotes the local ϵ-factor at $\mathfrak{p}$. This factor depends on the choice of a local Haar measure and an additive character at p (see [14] for details). We choose the Haar measure dx which gives $\mathbb{Z}_p$ measure 1, and the additive character $\tau : (\mathbb{Q}_p,+) \to \mathbb{C}^\times$ given by $\tau(ap^{-m}) = \exp(2\pi i a/p^m)$ with $a \in \mathbb{Z}_p$ (these are the choices used in [3]).

It is vital to know the p-integrality of the above L-function when tackling Kato's higher congruences. We now put $a_n = \mathbf{L}_p(E,\rho_n)$, and consider both

$$b_n = a_n/N_{0,n}(a_0) \quad \text{and} \quad c_n = b_n/\phi(b_{n-1}).$$

Kato's calculations in [9] of the image of K_1 predict that

$$\prod_{1\leqslant i\leqslant n} N_{i,n}(c_i)^{p^i} \equiv 1 \mod p^{2n}.$$

THEOREM 1.2. — *Under the same hypotheses as the previous result, the non-abelian congruences*

$$\prod_{1\leqslant i\leqslant n} N_{i,n}(c_i)^{p^i} \equiv 1 \mod p^{n+1}$$

hold true.

When $n > 1$ these congruences are (conjecturally) not the best possible. For example, if $n = 2$ we expect a congruence modulo p^4 not p^3, if $n = 3$ we expect one modulo p^6 not p^4, and so on. However when $n = 1$, we have proved the congruences found numerically by Tim and Vladimir Dokchitser.

THEOREM 1.3. — *Under the same hypotheses, the congruences computed in* [5] *always hold, i.e.*

$$a_1 \equiv N_{0,1}(a_0) \mod p.$$

Remarks.

(i) Theorem 1.3 was proved at $p = 3$ by T. Bouganis in [1], using properties of the 3-adic Eisenstein measure. Likewise, he had a non-integral version of Theorem 1.1.

(ii) In fact Hypothesis (Per) is not actually necessary. However, it must then be replaced with the assumption that the ρ_n-twisted homology of the space of modular symbols on $\Gamma_0(N_E)$ is p-integral (which is an old conjecture of Glenn Stevens from [13]).

(iii) If we replace Hypothesis (Ord) with the condition that E has bad multiplicative reduction at p, then the congruences mentioned above still hold.

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2. Algebraic-valued distributions

Let $\rho_k := \mathrm{Ind}_{K_k}^{F_k}\,\chi_{\rho_k}$ denote the two-dimensional Artin representation over F_k, and write $\rho_k/F_n := \mathrm{Res}_{F_n}\,\rho_k$ for its restriction to F_n. Consider the finite set of primes

$$S = \{v : v \text{ is a prime of } F_n,\ v|p\Delta\}\,.$$

Our main goal is to show for every integer $n \geqslant k$, there exists a $\overline{\mathbb{Q}}$-valued distribution interpolating

$$\text{simple factors} \times \frac{L_S(1, E, \rho_k/F_n \otimes \psi^{-1})}{\text{a period}}$$

for a suitable family of Hecke characters ψ (see Theorem 3.4 for the precise statement).

2.1. Hilbert modular forms

Let F be a totally real field such that $F/\mathbb{Q}$ is abelian. Following the notation from [10], let $h = |Cl^{\dagger}(F)|$ be the narrow class number of F, and choose ideles $t_1, \ldots, t_h$ such that $\tilde{t}_\lambda \lhd \mathcal{O}_F$ (the ideals generated by the t_λ) are all prime to p, and form a complete set of representatives for $Cl^{\dagger}(F)$. We also denote the different of $F/\mathbb{Q}$ by $\mathfrak{d}_F$.

Hilbert automorphic forms over F are holomorphic functions $\mathbf{f} : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ satisfying certain automorphy properties (see [10] or [12] for details). They also correspond to h-tuples $(f_1, \ldots, f_h)$ of Hilbert modular forms on $\mathcal{H}^d$ (where $d = [F : \mathbb{Q}]$). If $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ (the set of Hilbert automorphic forms of parallel weight k, level $\mathfrak{c}$ and character ψ) then

$$f_\lambda|_k\gamma = \psi(\gamma) f_\lambda$$

for all $\gamma \in \Gamma_\lambda(\mathfrak{c})$, where

$$\Gamma_\lambda(\mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b \in \tilde{t}_\lambda^{-1}\mathfrak{d}_F^{-1}, c \in \tilde{t}_\lambda \mathfrak{c}\mathfrak{d}_F, a, d \in \mathcal{O}_F, ad - bc \in \mathcal{O}_F^{\times} \right\}.$$

We define

$$e_F(\xi z) = \exp\left(2\pi i \sum_{1 \leqslant a \leqslant d} \xi^{\tau_a} z_a\right)$$

where $z = (z_1, \ldots, z_d) \in \mathcal{H}^d$, $\xi \in F$ and $\tau_1, \ldots, \tau_d$ are the embeddings $F \hookrightarrow \mathbb{R}$. Then, each component f_λ has a Fourier expansion of the form

$$f_\lambda(z) = \sum_{\xi} a_\lambda(\xi) e_F(\xi z),$$

where the sum is taken over all totally positive $\xi \in \tilde{t}_\lambda$ and $\xi = 0$. If $\mathbf{f}$ is a cusp form, then $a_\lambda(0) = 0$ for all λ. The set of cusp forms of parallel weight k, level $\mathfrak{c}$ and character ψ is written $\mathcal{S}_k(\mathfrak{c}, \psi)$.

The form $\mathbf{f}$ itself also has Fourier coefficients $C(\mathfrak{m}, \mathbf{f})$ which satisfy

$$C(\mathfrak{m}, \mathbf{f}) = \begin{cases} a_\lambda(\xi) N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-k/2} & \text{if the ideal } \mathfrak{m} = \xi \tilde{t}_\lambda^{-1} \text{ is integral;} \\ 0 & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases}$$

We will use certain linear operators on the space of Hilbert automorphic forms. Let $\mathfrak{q}$ be an integral ideal of $\mathcal{O}_F$, and q an idele such that $\tilde{q} = \mathfrak{q}$. We define the operators $\mathfrak{q}$ and $U(\mathfrak{q})$ on $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$:

$$(\mathbf{f}|\mathfrak{q})(x) = N_{F/\mathbb{Q}}(\mathfrak{q})^{-k/2}\mathbf{f}\left(x\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$(\mathbf{f}|U(\mathfrak{q}))(x) = N_{F/\mathbb{Q}}(\mathfrak{q})^{k/2-1}\sum_{v\in\mathcal{O}_F/\mathfrak{q}}\mathbf{f}\left(x\begin{pmatrix} 1 & v \\ 0 & q \end{pmatrix}\right).$$

These operators may also be described by their effect on the Fourier coefficients of $\mathbf{f}$, namely

$$C(\mathfrak{m}, \mathbf{f}\big|\mathfrak{q}) = C(\mathfrak{m}\mathfrak{q}^{-1}, \mathbf{f}) \quad \text{and} \quad C(\mathfrak{m}, \mathbf{f}|U(\mathfrak{q})) = C(\mathfrak{m}\mathfrak{q}, \mathbf{f}).$$

We also use the involution $J_{\mathfrak{c}}$, which is defined by

$$(\mathbf{f}|J_{\mathfrak{c}})(x) = \psi(\det(x)^{-1})\mathbf{f}\left(x\begin{pmatrix} 0 & 1 \\ c_0 & 0 \end{pmatrix}\right)$$

where c_0 is an idele such that $\tilde{c}_0 = \mathfrak{c}\mathfrak{d}_F^2$. Then, $\mathbf{f}|J_{\mathfrak{c}} \in \mathcal{M}_k(\mathfrak{c}, \psi^{-1})$. This map has the property

$$\mathbf{f}|J_{\mathfrak{m}\mathfrak{c}} = N_{F/\mathbb{Q}}(\mathfrak{m})^{k/2}(\mathbf{f}\big|J_{\mathfrak{c}})\big|\mathfrak{m}.$$

Further, when $\mathbf{f}$ is a primitive form in $\mathcal{M}_k(\mathfrak{c}, \psi)$, we have

$$\mathbf{f}|J_{\mathfrak{c}} = \Lambda(\mathbf{f})\mathbf{f}^{\iota},$$

where $\Lambda(\mathbf{f})$ is a root of unity, and $\mathbf{f}^{\iota}$ is the form defined by $C(\mathfrak{m}, \mathbf{f}^{\iota}) = \overline{C(\mathfrak{m}, \mathbf{f})}$.

Remark. — If $f_E \in \mathcal{S}_2^{\mathrm{new}}(\Gamma_0(N_E))$ is the newform associated to $E_{/\mathbb{Q}}$, then we write $\mathbf{f}$ for the base change of f_E to the totally real field F. Assuming $F/\mathbb{Q}$ is abelian, this is the Hilbert automorphic form whose L-series satisfies

$$L(s, \mathbf{f}) = \prod_{\psi\in\hat{G}} L(s, f_E, \psi)$$

where $G = \mathrm{Gal}(F/\mathbb{Q})$.

We introduce the following notation: for a character $\chi\colon \mathrm{Gal}(\mathbb{Q}_{FT}/F_n) \to \mathbb{C}^\times$, we write $\chi^\dagger : \mathcal{I}_{F_n} \to \mathbb{C}^\times$ for the character of ideals associated to χ via composition with the reciprocity map. Specifically $\chi^\dagger$ is normalised by

$$\chi^\dagger(\mathfrak{q}) = \chi(\mathrm{Frob}_{\mathfrak{q}})$$

for all primes $\mathfrak{q}$ of F_n, where $\mathrm{Frob}_{\mathfrak{q}}$ denotes an arithmetic Frobenius element at $\mathfrak{q}$.

Now, let K/F be a totally imaginary quadratic extension. We have the following theorem due to Serre [11]:

THEOREM 2.1. — *If ρ is an Artin representation over F which is induced from the Hecke character χ_ρ over K, then there exists a Hilbert automorphic form $\mathbf{g}_\rho$ over F such that $\mathbf{g}_\rho \in \mathcal{S}_1(\mathfrak{c}(\mathbf{g}_\rho), (\det \rho)^\dagger)$ and*

$$L(s, \mathbf{g}_\rho) = L(s, \rho).$$

Further, $\mathbf{g}_\rho$ is primitive if and only if χ_ρ is a primitive character.

It is easily checked that the Fourier coefficients of $\mathbf{g}_\rho$ are

$$C(\mathfrak{m}, \mathbf{g}_\rho) = \sum_{\substack{\mathfrak{a} \lhd \mathcal{O}_K, \\ \mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{m}}} \chi_\rho^\dagger(\mathfrak{a}).$$

Also, in the case $F = F_k, K = K_k$ and $\rho = \rho_k$, we assume $\gcd(p\Delta, N_E) = 1$ which implies that $\mathfrak{c}(\mathbf{f})$ and $\mathfrak{c}(\mathbf{g}_{\rho_k})$ are coprime ideals of $\mathcal{O}_{F_k}$.

The character $(\det \rho)^\dagger$ can be written as

$$(\det \rho)^\dagger(\mathfrak{a}) = \theta_{K/F}(\mathfrak{a}) \chi_\rho^\dagger(\mathfrak{a}\mathcal{O}_K)$$

where $\theta_{K/F}$ is the quadratic character of K/F, given on primes of $\mathcal{O}_F$ by

$$\theta_{K/F}(\mathfrak{q}) = \begin{cases} 1 & \text{if } \mathfrak{q} \text{ splits in } K/F \\ -1 & \text{if } \mathfrak{q} \text{ is inert in } K/F \\ 0 & \text{if } \mathfrak{q} \text{ ramifies in } K/F. \end{cases}$$

We use a non-standard normalisation of the Petersson inner product (from [10]), namely

$$\langle \mathbf{F}, \mathbf{G} \rangle_{\mathfrak{c}} := \sum_{\lambda=1}^{h} \int_{\Gamma_\lambda(\mathfrak{c}) \backslash \mathcal{H}^d} \overline{\mathbf{F}_\lambda(z)} \mathbf{G}_\lambda(z) N(y)^k d\nu(z)$$

where $d = [F : \mathbb{Q}]$, and

$$d\nu(z) = \prod_{1 \leqslant j \leqslant d} y_j^{-2} dx_j dy_j.$$

Finally $\Omega_{E/F_n}^{\mathrm{Aut}} = (2\pi)^{\phi(p^n)} \left\langle \mathbf{f}_{/F_n}, \mathbf{f}_{/F_n} \right\rangle_{\mathfrak{c}(\mathbf{f})}$ denotes the automorphic period.

2.2. Integrality

We will study the value at $s = 1$ of the normalised Rankin-Selberg product

$$\Psi(s, \mathbf{f}, \mathbf{g}_\rho) = \left(\frac{\Gamma(s)}{(2\pi)^s} \right)^{2[F:\mathbb{Q}]} L_{\mathfrak{c}}(2s-1, (\det \rho)^\dagger) L(s, \mathbf{f}, \mathbf{g}_\rho)$$

where $\mathfrak{c} = \mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g}_\rho)$, and

$$L(s, \mathbf{f}, \mathbf{g}_\rho) = \sum_{\mathfrak{a}} C(\mathfrak{a}, \mathbf{f}) C(\mathfrak{a}, \mathbf{g}_\rho) N_{F/\mathbb{Q}}(\mathfrak{a})^{-s}.$$

Our first goal is to prove that

$$\epsilon_F(0, \rho) \cdot \frac{\Psi(1, \mathbf{f}, \mathbf{g}_\rho^\iota)}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}} \in \mathcal{O}_{\mathbb{C}_p},$$

i.e. that this quantity is p-integral (it is already known to be algebraic by results of Shimura et al).

We will need a few preparatory lemmas, starting with a result about the epsilon factor $\epsilon_F(s, \rho)$. The Artin L-function $L(s, \rho)$ obeys the functional equation

$$\Gamma_\infty(s) L(s, \rho) = \epsilon_F(s, \rho) \Gamma_\infty(1 - s) L(1 - s, \rho^\vee)$$

where $\rho^\vee$ is the contragredient representation, and

$$\Gamma_\infty(s) := ((2\pi)^{-s}\Gamma(s))^{[F:\mathbb{Q}]}.$$

The global ϵ-factor at zero may be decomposed into an infinite product

$$\epsilon_F(0, \rho) = \prod_{\text{all places } v} \epsilon_{F_v}(\rho_v, \psi_\nu, dx_\nu)$$

where each local factor depends on the normalisation of additive characters ψ_ν, and Haar measures dx_ν (however the product does not).

Lemma 2.2. — *Setting* $\epsilon_F(\rho) = \epsilon_F(0, \rho)$, *we have*

$$\Lambda(\mathbf{g}_\rho) = i^{-[F:\mathbb{Q}]} N_{F/\mathbb{Q}}(\mathfrak{c}\mathfrak{d}_F^2)^{-1/2} \epsilon_F(\rho).$$

Proof. — Following Shimura in [12], we define

$$R(s, \mathbf{g}) := N_{F/\mathbb{Q}}(\mathfrak{c}\mathfrak{d}_F^2)^{s/2} \Gamma_\infty(s) L(s, \mathbf{g})$$

where $\mathbf{g}$ is a Hilbert automorphic form of parallel weight 1 and conductor $\mathfrak{c}$. Then from [12], (2.48) there is a functional equation

$$R(s, \mathbf{g}) = i^{[F:\mathbb{Q}]} R(1 - s, \mathbf{g}|J_{\mathfrak{c}}).$$

Supposing $\mathbf{g}$ is primitive, we have $\mathbf{g}|J_{\mathfrak{c}} = \Lambda(\mathbf{g})\mathbf{g}^\iota$, so the functional equation becomes $R(s, \mathbf{g}) = i^{[F:\mathbb{Q}]} \Lambda(\mathbf{g}) R(1-s, \mathbf{g}^\iota)$. However, taking $\mathbf{g} = \mathbf{g}_\rho$ we obtain $L(s, \rho) = L(s, \mathbf{g})$ and $L(s, \rho^\vee) = L(s, \mathbf{g}^\iota)$. Therefore

$$\Gamma_\infty(s) L(s, \rho) = \epsilon_F(s, \rho) \Gamma_\infty(1 - s) L(1 - s, \rho^\vee)$$

can be rewritten as

$$R(s, \mathbf{g}) = \epsilon_F(s, \rho) N_{F/\mathbb{Q}}(\mathfrak{c}\mathfrak{d}_F^2)^{s-1/2} R(1 - s, \mathbf{g}^\iota).$$

Comparing this with the functional equation from [12], it follows that there is an equality $i^{[F:\mathbb{Q}]}\Lambda(\mathbf{g}) = \epsilon_F(s,\rho)N_{F/\mathbb{Q}}((\mathfrak{c}\mathfrak{d}_F^2)^{s-1/2}$, which gives the result. □

We use the following integral representation, a special case of [12], (4.32):

Proposition 2.3.

$$\Psi(1,\mathbf{f},\mathbf{g}^\iota) = D_F^{1/2}\pi^{-[F:\mathbb{Q}]}\left\langle \mathbf{f}^\iota, V(0)\right\rangle_{\mathfrak{c}}$$

where D_F is the discriminant of $F/\mathbb{Q}$ and

$$V(0) = \mathbf{g}^\iota \cdot K_1^0(0;\mathfrak{c},\mathcal{O}_F;(\det\rho)^{\dagger-1}),$$

with K_1^0 the Eisenstein series given in [10] *equation* (4.5) *whose λ-components are:*

$$K_1^0(0;\mathfrak{c},\mathcal{O}_F;\omega)_\lambda(z) \\ = N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{1/2}\sum_{c,d}\operatorname{sign}(N_{F/\mathbb{Q}}(d))\omega^*(d\mathcal{O}_F)N_{F/\mathbb{Q}}(cz+d)^{-1}.$$

Note that the sum is taken over the set of equivalence classes

$$(c,d)\in\frac{\tilde{t}_\lambda\mathfrak{d}_F\mathfrak{c}\times\mathcal{O}_F}{\sim}$$

where the relation $\sim$ is defined by $(c,d)\sim(uc,ud)$ for all $u\in\mathcal{O}_F^\times$.

It is useful to convert K_1^0 to an Eisenstein series which has a user-friendly Fourier expansion; we can do this via the involution $J_{\mathfrak{c}}$. Using [10] (4.6), one can show:

$$K_1^0(0;\mathfrak{c},\mathcal{O}_F;(\det\rho)^{\dagger-1})\big|J_{\mathfrak{c}} = \frac{(4\pi i)^{[F:\mathbb{Q}]}}{D_F^{1/2}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{g})\mathfrak{d}_F^2)^{1/2}}E_1(0,\mathfrak{c},(\det\rho)^{\dagger-1}).$$

Here E_1 is the Eisenstein series (4.13) in [10], with λ-components

$$E_1(0,\mathfrak{c},\omega)_\lambda(z) \\ = \frac{N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-1/2}D_F^{1/2}}{(-4\pi i)^{[F:\mathbb{Q}]}}\sum_{c,d}\operatorname{sign}(N_{F/\mathbb{Q}}(c))\omega^*(c\mathcal{O}_F)N_{F/\mathbb{Q}}(cz+d)^{-1}$$

such that ω is an ideal character modulo $\mathfrak{c}$, and the sum ranges over

$$(c,d)\in\frac{\mathcal{O}_F\times\tilde{t}_\lambda^{-1}\mathfrak{d}_F^{-1}}{\sim}.$$

The Fourier expansion of each λ-component is computed in [10], Prop 4.2:

$$E_1(0,\mathfrak{c},(\det\rho)^{\dagger-1})_\lambda(z) = N_{F/\mathbb{Q}}(\tilde{t}_\lambda)^{-1/2}\sum_{0\ll\xi\in\tilde{t}_\lambda}a_\lambda(\xi)e_F(\xi z)$$

with

$$a_\lambda(\xi) = \sum_{\substack{\tilde{\xi}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_F,\\ b\in\tilde{t}_\lambda}} (\det\rho)^{\dagger-1}(\tilde{c}).$$

We are now in a good position to prove our integrality result.

THEOREM 2.4. — *Let $\mathbf{g} = \mathbf{g}_\rho$. If there exists no non-trivial congruence modulo $\mathfrak{p}$ between $\mathbf{f}$ and another automorphic form in $\mathcal{M}_2(\mathfrak{c}(\mathbf{f}))$, then*

$$\epsilon_F(\rho)\cdot\frac{\Psi(1,\mathbf{f},\mathbf{g}^\iota)}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}$$

is p-integral, where $\epsilon_F(\rho) = \epsilon_F(0,\rho)$ as before.

Proof. — Let $\mathfrak{c} = \mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})$. By Proposition 2.3,

$$\Psi(1,\mathbf{f},\mathbf{g}^\iota) = D_F^{1/2}\pi^{-[F:\mathbb{Q}]}\langle\mathbf{f}^\iota, V(0)\rangle_{\mathfrak{c}}$$

where $V(0) = \mathbf{g}^\iota\cdot K_1^0(0;\mathfrak{c},\mathcal{O}_F;(\det\rho)^{\dagger-1})$. Consider $V(0)\big|J_{\mathfrak{c}}$; by our earlier formula for $K_1^0\big|J_{\mathfrak{c}}$ we have

$$\begin{aligned}V(0)|J_{\mathfrak{c}} &= (\mathbf{g}^\iota\big|J_{\mathfrak{c}})\cdot(K_1^0\big|J_{\mathfrak{c}})\\ &= \Lambda(\mathbf{g}^\iota)N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{f}))^{1/2}(\mathbf{g}\big|\mathfrak{c}(\mathbf{f}))\cdot(K_1^0\big|J_{\mathfrak{c}})\\ &= \Lambda(\mathbf{g}^\iota)(4\pi i)^{[F:\mathbb{Q}]}D_F^{-1/2}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{f}))^{1/2}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{g})\mathfrak{d}_F^2)^{-1/2}(\mathbf{g}\big|\mathfrak{c}(\mathbf{f}))\\ &\qquad\cdot E_1(0,\mathfrak{c},(\det\rho)^{\dagger-1}).\end{aligned}$$

We now employ the trace map $\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}(\mathbf{f})} : \mathcal{M}_2(\mathfrak{c},\psi)\to\mathcal{M}_2(\mathfrak{c}(\mathbf{f}),\psi)$, which is defined by

$$\left(\mathbf{H}\big|\,\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}(\mathbf{f})}\right)(x) = \sum_{v\in T}\mathbf{H}\left(x\begin{pmatrix}1 & 0\\ cv & 1\end{pmatrix}\right).$$

where c is an idele such that $\tilde{c} = \mathfrak{c}(\mathbf{f})$, and T is a set of coset representatives for $\mathcal{O}_F/\mathfrak{c}(\mathbf{g})$. This map has the property

$$\langle\mathbf{F},\mathbf{H}\rangle_{\mathfrak{c}} = \left\langle\mathbf{F},\mathbf{H}\big|\,\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})}$$

for any two Hilbert automorphic forms $\mathbf{H}\in\mathcal{M}_2(\mathfrak{c},\psi), \mathbf{F}\in\mathcal{S}_2(\mathfrak{c}(\mathbf{f}),\psi)$. Further, from [10] equation (4.11) we have the formula

$$\mathbf{H}\big|\,\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}(\mathbf{f})} = \mathbf{H}\big|J_{\mathfrak{c}}\circ U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}.$$

This arises from the definitions of the operators, and the matrix identity

$$\begin{pmatrix}1 & 0\\ cv & 1\end{pmatrix} = (cm)^{-1}\begin{pmatrix}0 & 1\\ cm & 0\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & m\end{pmatrix}\begin{pmatrix}0 & 1\\ c & 0\end{pmatrix}$$

which holds for any c, m and v.

Remark. — Setting $\Theta = (\mathbf{g}|\mathfrak{c}(\mathbf{f})) \cdot E_1(0, (\det\rho)^{\dagger-1})$, we calculate

$$\begin{aligned}\Psi(1,\mathbf{f},\mathbf{g}^\iota) &= D_F^{1/2}\pi^{-[F:\mathbb{Q}]}\left\langle \mathbf{f}^\iota, V(0)\right\rangle_{\mathfrak{c}} \\ &= D_F^{1/2}\pi^{-[F:\mathbb{Q}]}\left\langle \mathbf{f}^\iota, V(0)\middle|\,\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})} \\ &= D_F^{1/2}\pi^{-[F:\mathbb{Q}]}\left\langle \mathbf{f}^\iota, V(0)\middle| J_{\mathfrak{c}}\circ U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})} \\ &= \Lambda(\mathbf{g}^\iota)(4i)^{[F:\mathbb{Q}]}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{f}))^{1/2}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{g})\mathfrak{d}_F^2)^{-1/2} \\ &\qquad\qquad\qquad \cdot\left\langle \mathbf{f}^\iota, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})}.\end{aligned}$$

Observe that $\Lambda(\mathbf{g}^\iota)(4i)^{[F:\mathbb{Q}]}N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{f}))^{1/2}$ is a p-adic unit, as $\gcd(\mathfrak{p}, 4\mathfrak{c}(\mathbf{f}))=1$. Also, from Lemma 2.2 we know that $\mathrm{ord}_{\mathfrak{p}}(N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{g})\mathfrak{d}_F^2)^{1/2}) = \mathrm{ord}_{\mathfrak{p}}(\epsilon_F(\rho))$. Therefore,

$$\mathrm{ord}_{\mathfrak{p}}\left(\epsilon_F(\rho)\cdot\frac{\Psi(1,\mathbf{f},\mathbf{g}^\iota)}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}\right) = \mathrm{ord}_{\mathfrak{p}}\left(\frac{\left\langle \mathbf{f}^\iota, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}\right)$$

so it suffices to prove the p-integrality of the quantity on the right hand side. As the operator J is an involution,

$$\begin{aligned}\left\langle \mathbf{f}^\iota, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})} &= \left\langle \mathbf{f}^\iota\middle| J_{\mathfrak{c}(\mathbf{f})}, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\right\rangle_{\mathfrak{c}(\mathbf{f})} \\ &= \Lambda(\mathbf{f}^\iota)\left\langle \mathbf{f}, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\right\rangle_{\mathfrak{c}(\mathbf{f})}.\end{aligned}$$

By choosing a basis for the space $\mathcal{M}_2(\mathfrak{c}(\mathbf{f}))$ which includes $\mathbf{f}$, we may express

$$\Theta\big|U(\mathfrak{c}(\mathbf{g})) = c\mathbf{f} + \sum_{\mathbf{f}_i\neq\mathbf{f}} c_i\mathbf{f}_i\big|\mathfrak{b}_i$$

for algebraic numbers c_i (which are almost all zero), and primitive forms $\mathbf{f}_i$ of level $\mathfrak{a}_i$ such that $\mathfrak{a}_i\mathfrak{b}_i$ divides $\mathfrak{c}(\mathbf{f})$.

We deduce that

$$\frac{\left\langle \mathbf{f}^\iota, \Theta\middle| U(\mathfrak{c}(\mathbf{g}))\circ J_{\mathfrak{c}(\mathbf{f})}\right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} = \Lambda(\mathbf{f}^\iota)\left(c + \sum_{\mathbf{f}_i\neq\mathbf{f}} c_i\frac{\left\langle\mathbf{f},\mathbf{f}_i\middle|\mathfrak{b}_i\right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}\right).$$

Quoting Dünger's paper [7], Section 5.5,

$$\frac{\left\langle\mathbf{f},\mathbf{f}_i\middle|\mathfrak{b}_i\right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}_i\rangle_{\mathfrak{c}(\mathbf{f})}} = \left(\frac{L(s,\mathbf{f},\mathbf{f}_i^\iota|\mathfrak{b}_i)}{L(s,\mathbf{f},\mathbf{f}_i^\iota)}\right)_{s=2}.$$

One may therefore write

$$\frac{\left\langle\mathbf{f},\mathbf{f}_i\middle|\mathfrak{b}_i\right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} = \left(\frac{L(s,\mathbf{f},\mathbf{f}_i^\iota|\mathfrak{b}_i)}{L(s,\mathbf{f},\mathbf{f}_i^\iota)}\right)_{s=2}\times\frac{\langle\mathbf{f},\mathbf{f}_i\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}.$$

But $\langle \mathbf{f}, \mathbf{f}_i \rangle_{\mathfrak{c}(\mathbf{f})} = 0$ as $\mathbf{f}$ and $\mathbf{f}_i$ are distinct primitive forms, whence

$$\frac{\left\langle \mathbf{f}, \mathbf{f}_i \middle| \mathfrak{b}_i \right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}} = 0$$

for each i. As a consequence,

$$\frac{\left\langle \mathbf{f}^{\iota}, \Theta | U(\mathfrak{c}(\mathbf{g})) \circ J_{\mathfrak{c}(\mathbf{f})} \right\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}} = \Lambda(\mathbf{f}^{\iota}) c.$$

Lastly $\Lambda(\mathbf{f}^{\iota})$ is a root of unity, thus it suffices to prove c is p-integral. Suppose not; then $c^{-1} \equiv 0 \mod \mathfrak{p}$. We know that both $E_1(0, (\det \rho)^{\dagger -1})$ and $\mathbf{g}_\rho$ have p-integral Fourier coefficients, so $\Theta = (\mathbf{g}_\rho | \mathfrak{c}(\mathbf{f})) \cdot E_1(0, \mathfrak{c}, (\det \rho)^{\dagger -1})$ does too. Therefore

$$c^{-1} \Theta \big| U \equiv 0 \mod \mathfrak{p},$$

whence

$$\begin{aligned} \mathbf{f} &= c^{-1} \Theta \big| U - \sum_{\mathbf{f}_i \neq \mathbf{f}} c^{-1} c_i \mathbf{f}_i \big| \mathfrak{b}_i \\ &\equiv - \sum_{\mathbf{f}_i \neq \mathbf{f}} c^{-1} c_i \mathbf{f}_i \big| \mathfrak{b}_i \mod \mathfrak{p}. \end{aligned}$$

It follows that $\mathbf{f}$ is congruent modulo $\mathfrak{p}$ to some other automorphic form in $\mathcal{M}_2(\mathfrak{c}(\mathbf{f}))$, contradicting the hypothesis of the theorem. This completes the proof. □

2.3. Constructing the distribution

Having established our integrality result, we can now go on to construct the distribution. First we work over a totally real field F.

For a finite place $v \neq \mathfrak{p}$ of F, we label roots $\alpha(v)$, $\alpha'(v)$ of the polynomial

$$X^2 - C(v, \mathbf{f}) X + N_{F/\mathbb{Q}}(v) = (X - \alpha(v))(X - \alpha'(v)).$$

We also define $\alpha(\mathfrak{p})$ and $\alpha'(\mathfrak{p})$ as the roots of

$$X^2 - C(\mathfrak{p}, \mathbf{f}) X + p = (X - \alpha(\mathfrak{p}))(X - \alpha'(\mathfrak{p})).$$

where $\alpha(\mathfrak{p})$ is the $\mathfrak{p}$-adic unit, and $\alpha'(\mathfrak{p})$ is the non-unit root. From these definitions, we extend $\alpha(\mathfrak{m})$, $\alpha'(\mathfrak{m})$ multiplicatively to all ideals $\mathfrak{m}$ of $\mathcal{O}_F$.

Definition 2.5. — *Set $\mathfrak{l}_0 := \prod_{\mathfrak{q} | \Delta} \mathfrak{q}$. Then the $\mathfrak{p}\mathfrak{l}_0$-stabilisation of $\mathbf{f}$ is defined to be*

$$\mathbf{f}_0 := \sum_{\mathfrak{a} | \mathfrak{p}\mathfrak{l}_0} M(\mathfrak{a}) \alpha'(\mathfrak{a}) \cdot \mathbf{f} \big| \mathfrak{a}$$

where M is the Möbius function on ideals.

Following (3.14) of [10], we also define

$$\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0} := \sum_{\mathfrak{n}|\mathfrak{p}\mathfrak{l}_0} M(\mathfrak{n}) \cdot \mathbf{g}_\rho \big| U(\mathfrak{n}) \circ \mathfrak{n}.$$

In particular $\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0} \in \mathcal{M}_1(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2, (\det\rho)^\dagger)$.

Set $\mathfrak{c}_0 = \mathfrak{p}\mathfrak{l}_0\mathfrak{c}(\mathbf{f})$. We shall choose ideals $\mathfrak{m}'$ and $\mathfrak{l}'$ such that $\mathfrak{m}'$ is a power of $\mathfrak{p}$, $\mathrm{supp}(\mathfrak{l}') = \mathrm{supp}(\mathfrak{l}_0)$, and that $\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2|\mathfrak{m}'\mathfrak{l}'$. Clearly

$$\mathbf{f}_0 \in \mathcal{S}_{\mathbf{2}}(\mathfrak{c}_0) \subset \mathcal{S}_{\mathbf{2}}(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}')$$

and

$$\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0} \in \mathcal{M}_{\mathbf{1}}(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2, (\det\rho)^\dagger) \subset \mathcal{M}_{\mathbf{1}}(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}', (\det\rho)^\dagger).$$

Then the contragredient Euler factor is given by

$$\begin{aligned}\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho^\vee, s) := \prod_{v|\mathfrak{p}\mathfrak{l}_0} &(1-\alpha'(v)\hat{\beta}(v)N(v)^{-s})(1-\alpha'(v)\hat{\beta}'(v)N(v)^{-s}) \\ &\times (1-\alpha^{-1}(v)\beta(v)N(v)^{s-1})(1-\alpha^{-1}(v)\beta'(v)N(v)^{s-1}).\end{aligned}$$

N.B. We have factorised the Hecke polynomial as

$$X^2 - C(v,\mathbf{g}_\rho)X + (\det\rho)^\dagger(v) = (X-\beta(v))(X-\beta'(v)),$$

and similarly the dual Hecke polynomial via

$$X^2 - \overline{C(v,\mathbf{g}_\rho)}X + {(\det\rho)^\dagger}^{-1}(v) = (X-\hat{\beta}(v))(X-\hat{\beta}'(v)).$$

Remark. — Because we assumed that E is semistable over $\mathbb{Q}$, the coefficient

$$C(\mathfrak{c}(\mathbf{f}),\mathbf{f}) = (-1)^{\#\mathcal{T}_F^{ns}}$$

where $\mathcal{T}_F^{ns}$ denotes the set of finite places where E has non-split multiplicative reduction. In particular, $C(\mathfrak{c}(\mathbf{f}),\mathbf{f}) \neq 0$.

Lemma 2.6.

$$\begin{aligned}\Psi(s,\mathbf{f}_0,\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}) = N_{F/\mathbb{Q}}\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)}\right)^{1/2-s} &\Lambda(\mathbf{g}_\rho)\alpha\left(\frac{\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)}\right) \\ \times C(\mathfrak{c}(\mathbf{f}),\mathbf{f})\,&\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho^\vee,s)\Psi(s,\mathbf{f},\mathbf{g}_\rho^\iota).\end{aligned}$$

Proof. — Recall the formula $\mathbf{f}|J_{\mathfrak{m}\mathfrak{c}} = N_{F/\mathbb{Q}}(\mathfrak{m})^{k/2}(\mathbf{f}\big|J_\mathfrak{c})\big|\mathfrak{m}$ from Section 2.1. Since $\mathfrak{c}(\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}) = \mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2$, it follows that

$$\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'} = N_{F/\mathbb{Q}}\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)^{1/2} \cdot \left(\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)\Big|\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}.$$

For brevity, we will write

$$\mathbf{h} = \mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}.$$

Then

$$\begin{aligned}\Psi(s,&\mathbf{f}_0, \mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'})\\ &= N_{F/\mathbb{Q}}\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)^{1/2}\Psi\left(s,\mathbf{f}_0,\mathbf{h}\bigg|\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)\\ &= N_{F/\mathbb{Q}}\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)^{1/2-s}\Psi\left(s,\mathbf{f}_0\bigg|U\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right),\mathbf{h}\right)\\ &= N_{F/\mathbb{Q}}\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)^{1/2-s}\alpha\left(\frac{\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)C(\mathfrak{c}(\mathbf{f}),\mathbf{f})\Psi\left(s,\mathbf{f}_0,\mathbf{h}\right).\end{aligned}$$

Here we have exploited the fact that

$$L\left(s,\mathbf{f}_0,\mathbf{g}_\rho^\iota|\mathfrak{a}\right) = N_{F/\mathbb{Q}}(\mathfrak{a})^{-s}L\left(s,\mathbf{f}_0|U(\mathfrak{a}),\mathbf{g}_\rho^\iota\right)$$

for any ideal $\mathfrak{a}$, and also the formula

$$\mathbf{f}_0\big|U\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right) = \alpha\left(\frac{\mathfrak{m}'\mathfrak{l}'}{\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{p}^2\mathfrak{l}_0^2}\right)C(\mathfrak{c}(\mathbf{f}),\mathbf{f})\mathbf{f}_0$$

which follows by construction of the $\mathfrak{p}\mathfrak{l}_0$-stabilisation $\mathbf{f}_0$. Furthermore,

$$\Psi(s,\mathbf{f}_0,\mathbf{h}) = N_{F/\mathbb{Q}}(\mathfrak{p}^2\mathfrak{l}_0^2)^{1-2s}\alpha(\mathfrak{p}^2\mathfrak{l}_0^2)\Lambda(\mathbf{g}_\rho)\,\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho^\vee,s)\Psi(s,\mathbf{f},\mathbf{g}_\rho^\iota).$$

Combining the two equations together yields the required result. □

Definition 2.7. — *We define the complex linear functional $\mathcal{L}_F$ on the vector space $\mathcal{M}_{\mathbf{2}}(\mathfrak{c}_0)$ by the rule*

$$\mathcal{L}_F:\quad \Theta\longmapsto\frac{\left\langle\mathbf{f_0}^\iota,\Theta\big|J_{\mathfrak{c}_0}\right\rangle_{\mathfrak{c}_0}}{\left\langle\mathbf{f},\mathbf{f}\right\rangle_{\mathfrak{c}(\mathbf{f})}}.$$

We shall now consider the automorphic form

$$\Phi = \Phi(\rho/F,\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}') := \mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\cdot E_1(0,\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}',(\det\rho)^{\dagger-1})$$

where E_1 refers to the Eisenstein series of Section 2.2.

Corollary 2.8.

$$\begin{aligned}\frac{N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{d}_F^2)^{1/2}\Lambda(\mathbf{g}_\rho)}{\alpha(\mathfrak{c}(\mathbf{g}_\rho))}\quad&\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho^\vee,1)\frac{\Psi(1,\mathbf{f},\mathbf{g}_\rho^\iota)}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}\\ &=\frac{(-4i)^{[F:\mathbb{Q}]}}{\alpha(\mathfrak{m}'\mathfrak{l}')C(\mathfrak{c}(\mathbf{f}),\mathbf{f})}\mathcal{L}_F\left(\Phi\big|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right).\end{aligned}$$

Proof. — Applying the integral representation from Proposition 2.3, then using the trace map $\mathrm{Tr}_{\mathfrak{c}_0}^{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'}$ as we did to prove integrality, one obtains

$$\frac{\Psi(1,\mathbf{f}_0,\mathbf{g}_{\rho,\mathfrak{p}\mathfrak{l}_0}\big|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'})}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} = \frac{(-4i)^{[F:\mathbb{Q}]}}{N_{F/\mathbb{Q}}(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{l}'\mathfrak{d}_F^2)^{1/2}}\quad\mathcal{L}_F\left(\Phi\big|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right).$$

Combining this formula with Lemma 2.6 gives the desired result. □

Remark. — The left-hand side of the equation in the corollary does not depend on $\mathfrak{m}'$ or $\mathfrak{l}'$, so neither does the right hand side. As a consequence

$$\frac{(-4i)^{[F:\mathbb{Q}]}}{N_{F/\mathbb{Q}}(\mathfrak{d}_F)\alpha(\mathfrak{m}'\mathfrak{l}')C(\mathfrak{c}(\mathbf{f}),\mathbf{f})} \quad \mathcal{L}_F\left(\Phi\middle|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right)$$

will satisfy the axioms of a distribution, with respect to the finitely additive functions ψ.

We now consider the Hilbert automorphic form $\mathbf{g}_{\rho_k/F_n}$, the base change of $\mathbf{g}_{\rho_k}$ to F_n. Just as $\mathbf{g}_{\rho_k}$ is associated to $\rho_k = \operatorname{Ind}\chi_{\rho_k}$, we now show that $\mathbf{g}_{\rho_k/F_n}$ is also associated to an induced representation.

LEMMA 2.9. — *If $\rho_k = \operatorname{Ind}_{K_k}^{F_k}(\chi_{\rho_k})$, then*

$$L(s,\mathbf{g}_{\rho_k/F_n}) = L(s,\operatorname{Ind}_{K_n}^{F_n}(\operatorname{Res}_{K_n}\chi_{\rho_k})).$$

Proof. — Firstly, by the properties of the base change

$$L(s,\mathbf{g}_{\rho_k/F_n}) = \prod_{\psi\in\hat{G}} L(s,\rho_k\otimes\psi) = L(s,\rho_k\otimes R_{F_n/F_k})$$

where $G = \operatorname{Gal}(F_n/F_k)$, and $R_{F_n/F_k} = \operatorname{Ind}_{F_n}^{F_k}\mathbf{1}$ denotes its regular representation. However, the Artin formalism implies

$$L(s,\rho/M) = L(s,\operatorname{Ind}\rho/L)$$

whenever ρ is an Artin representation over M, and L is a subfield of M. Therefore

$$L(s,\rho_k\otimes R_{F_n/F_k}) = L(s,\rho_k\otimes\operatorname{Ind}_{F_n}^{F_k}\mathbf{1}) = L(s,\operatorname{Res}_{F_n}\rho_k\otimes\mathbf{1})$$

and the result follows because $\operatorname{Res}_{F_n}\rho_k = \operatorname{Ind}_{K_n}^{F_n}(\operatorname{Res}_{K_n}\chi_{\rho_k})$. □

N.B. In a slight abuse of notation, we have written ρ_k/F_n as shorthand for the Artin representation $\operatorname{Res}_{F_n}\rho_k = \operatorname{Ind}_{K_n}^{F_n}(\operatorname{Res}_{K_n}\chi_{\rho_k})$.

Let us denote by $\mathcal{G}_n$ the topological group $\operatorname{Gal}(F_{n,S}^{\mathrm{ab}}/F_n)$ where $F_{n,S}^{\mathrm{ab}}$ is the maximal abelian extension of F_n unramified outside the set $S = \{v : v|\mathfrak{p}\mathfrak{l}_0\}$ and the infinite places. For the remainder of this section, $\psi : \mathcal{G}_n \to \mathbb{C}^\times$ will be a character with conductor $\mathfrak{f}_\psi$. We shall apply our earlier results to $\mathbf{g}_\rho = \mathbf{g}_{\rho_k/F_n\otimes\psi}$, as the representation $\rho_k/F_n\otimes\psi$ is certainly induced by the character $\operatorname{Res}_{K_n}(\chi_{\rho_k}\otimes\psi)$ over K_n.

Further, it is easy to check that

$$\operatorname{Res}_{K_n}(\chi_{\rho_k}\otimes\psi)^\dagger = (\chi_{\rho_k}^\dagger\otimes\psi^\dagger)\circ N_{K_n/K_k}.$$

Also, the character of $\mathbf{g}_{\rho_k/F_n\otimes\psi}$ is $(\mathrm{Res}_{F_n}\det\rho_k)^\dagger\otimes\psi^{\dagger 2}$, and we have

$$(\mathrm{Res}_{F_n}(\det\rho_k)\otimes\psi^2)^\dagger = ((\det\rho_k)^\dagger\circ N_{F_n/F_k})\otimes\psi^{\dagger 2}.$$

Definition 2.10. — *The parallel weight 2 form $\Phi_\psi^{n,k}$ is given by*

$$\begin{aligned}\Phi_\psi^{n,k} &= \Phi_\psi^{n,k}(\rho_k/F_n\otimes\psi, \mathfrak{c}(\mathbf{f}_{/F_n})\mathfrak{m}'\mathfrak{l}') \\ &:= (\mathbf{g}_{\rho_k/F_n\otimes\psi,\mathfrak{p}\mathfrak{l}_0})\cdot E_1(0,\mathfrak{c}(\mathbf{f}_{/F_n})\mathfrak{m}'\mathfrak{l}', (\mathrm{Res}_{F_n}\det\rho_k)^{-1}\otimes\psi^{-2})\end{aligned}$$

where we now assume $\mathfrak{m}'$ and $\mathfrak{l}'$ satisfy $\mathfrak{c}(\mathbf{g}_\rho)(\mathfrak{p}\mathfrak{l}_0\mathfrak{f}_\psi)^2\big|\mathfrak{m}'\mathfrak{l}'$.

Applying Corollary 2.8 directly to $\mathbf{g}_{\rho_k/F_n\otimes\psi}$ produces

Corollary 2.11. — *For all $n\geqslant k$,*

$$\begin{aligned}\frac{(-4i)^{\phi(p^n)/2}}{\alpha(\mathfrak{m}'\mathfrak{l}')C(\mathfrak{c}(\mathbf{f}),\mathbf{f})}\mathcal{L}_{F_n}\left(\Phi_\psi^{n,k}\big|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right) \\ = \quad \frac{N_{F_n/\mathbb{Q}}(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n\otimes\psi})\mathfrak{d}_{F_n}^2)^{1/2}}{\alpha(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n\otimes\psi}))}\times\Lambda(\mathbf{g}_{\rho_k/F_n\otimes\psi}) \\ \times\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n\otimes\psi^{-1},1)\times\frac{\Psi(1,\mathbf{f}_{/F_n},\mathbf{g}^\iota_{\rho_k/F_n\otimes\psi})}{\left\langle\mathbf{f}_{/F_n},\mathbf{f}_{/F_n}\right\rangle_{\mathfrak{c}(\mathbf{f})}}.\end{aligned}$$

Furthermore, the Fourier coefficients of the λ-component of $\Phi_\psi^{n,k}$ are given by

$$\begin{aligned}\phi_{\psi,\lambda}^{n,k}(\xi) = \sum_{\xi=\xi_1+\xi_2}\sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_{K_n},\\ \mathfrak{a}\bar{\mathfrak{a}}=\xi_1\tilde{t}_\lambda^{-1}}}(\chi_{\rho_k}^\dagger\circ N_{K_n/K_k})(\mathfrak{a})\psi^\dagger(\xi_1\tilde{t}_\lambda^{-1}) \\ \times\sum_{\substack{\tilde{\xi_2}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_{F_n},\\ b\in\tilde{t}_\lambda}}((\det\rho_k)^\dagger\circ N_{F_n/F_k})^{-1}(\tilde{c})\psi^\dagger(\tilde{c})^{-2}.\end{aligned}$$

3. Congruences

The disadvantage of Corollary 2.11 is that it is hard to decipher exactly what it has to do with Kato's conjectures. We shall now interpret various quantites from this formula, back in terms of the arithmetic of $E_{/\mathbb{Q}}$.

3.1. The connection with elliptic curves

First, we will find an expression for the α-term from our main formula. Recall that we already made a choice of $\alpha(\mathfrak{q})$ for each $\mathfrak{q}$ dividing $\mathfrak{p}\mathfrak{l}_0$, in order to define the $\mathfrak{p}\mathfrak{l}_0$-stabilisation $\mathbf{f}_0$. For an Artin representation ρ over a field F, we denote its conductor by $\mathfrak{f}_\rho$. This is an ideal of $\mathcal{O}_F$, and we write $f(\rho,\mathfrak{q})$ for the exponent of the prime $\mathfrak{q}$ in $\mathfrak{f}_\rho$.

Lemma 3.1.

(i) *For each prime $q|p\Delta$, there exists a root α_q of $X^2 - a_q(E)X + q$ such that*

$$\alpha(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi)) = \alpha_p^{f(\rho_k/F_n\otimes\psi,\mathfrak{p})} \cdot \prod_{q|\Delta} \alpha_q^{\mathrm{ord}_q(A_{\tilde{\chi}})}$$

where $\tilde{\chi} = \mathrm{Res}_{K_n}(\chi_{\rho_k} \otimes \psi)$ and $A_{\tilde{\chi}} = N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})$.

(ii) *Furthermore, if we make the stronger assumption that ψ is a character of $\mathrm{Gal}(F^{\mathrm{ab}}_{n,\{\mathfrak{p}\}}/F_n)$ i.e. ψ is ramified only at the prime above p, then*

$$\mathrm{ord}_q(A_{\tilde{\chi}}) = p^n - p^{n-1} \quad \textit{for all } q|\Delta.$$

In particular, $\alpha(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi))$ is always a p-adic unit.

Proof. — For $\mathfrak{q} \neq \mathfrak{p}$, $\alpha(\mathfrak{q})$ is one of the eigenvalues of $\mathrm{Frob}_\mathfrak{q}$ acting on the Tate module $T_p(E)$. However, $\mathrm{Frob}_\mathfrak{q} = \mathrm{Frob}_q^{[f_{n,\mathfrak{q}}:\mathbb{F}_q]}$ where $f_{n,\mathfrak{q}}$ denotes the residue field of F_n at $\mathfrak{q}$. Therefore

$$\alpha(\mathfrak{q}) = \alpha_q^{[f_{n,\mathfrak{q}}:\mathbb{F}_q]}$$

for one of the roots α_q of $X^2 - a_q(E)X + q$.

For $\mathfrak{q} = \mathfrak{p}$, we instead consider $T_p(\tilde{E})$ where $\tilde{E}$ denotes the reduction of E over $f_{n,\mathfrak{p}}$. In this case, $\mathrm{rank}_{\mathbb{Z}_p} T_p(\tilde{E}) = 1$ because we have assumed E has good ordinary reduction at p, so $\alpha(\mathfrak{p})$ is the unique eigenvalue of $\mathrm{Frob}_\mathfrak{p}$ acting on $T_p(\tilde{E})$. Applying the same argument as above, $\alpha(\mathfrak{p}) = \alpha_p$.

Set $\mathfrak{c} = \mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi)$ for brevity. Then $\alpha(\mathfrak{c})$ is defined multiplicatively, so

$$\alpha(\mathfrak{c}) = \prod_{\substack{\mathfrak{q}\in\mathrm{Spec}(\mathcal{O}_{F_n})\\ \mathfrak{q}|\mathfrak{c}}} \alpha(\mathfrak{q})^{\mathrm{ord}_\mathfrak{q}(\mathfrak{c})} = \prod_{q|N_{F_n/\mathbb{Q}}(\mathfrak{c})} \alpha_q^{\mathrm{ord}_q(N_{F_n/\mathbb{Q}}(\mathfrak{c}))}.$$

Because $\rho_k/F_n \otimes \psi = \mathrm{Ind}_{K_n}^{F_n} \tilde{\chi}$, by [12], Section 5

$$\mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi) = N_{K_n/F_n}(\mathfrak{f}_{\tilde{\chi}})\mathrm{Disc}(K_n/F_n).$$

However $\mathrm{Disc}(K_n/F_n) = \mathfrak{p}$, which means

$$N_{F_n/\mathbb{Q}}(\mathfrak{c}) = p \cdot N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}}).$$

The primes dividing $N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})$ are those dividing $p\Delta$, whence

$$\alpha(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi)) = \alpha_p^{1+\mathrm{ord}_p(A_{\tilde{\chi}})} \cdot \prod_{q|\Delta} \alpha_q^{\mathrm{ord}_q(A_{\tilde{\chi}})}.$$

Let $\mathfrak{P}$ denote the unique prime of K_n above p. We have $N_{K_n/\mathbb{Q}}(\mathfrak{P}) = p$, from which we deduce

$$\begin{aligned} 1 + \mathrm{ord}_p(A_{\tilde{\chi}}) &= 1 + \mathrm{ord}_{\mathfrak{P}}(\mathfrak{f}_{\tilde{\chi}}) \\ &= f(\rho_k/F_n \otimes \psi, \mathfrak{p}) \end{aligned}$$

using standard results on Artin conductors. This proves (i).

It remains to prove assertion (ii). We now assume ψ is ramified only above p, and that $q|\Delta$. We then obtain

$$\mathrm{ord}_q(N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})) = \sum_{\mathfrak{Q}|q} f(\tilde{\chi}, \mathfrak{Q})[k_{n,\mathfrak{Q}} : \mathbb{F}_q]$$

where the sum is taken over the primes of K_n above q, and $k_{n,\mathfrak{Q}}$ denotes the residue field of K_n at $\mathfrak{Q}$. Under our additional assumption on ψ, we can say

$$f(\tilde{\chi}, \mathfrak{Q}) = f(\mathrm{Res}_{K_n}\, \chi_{\rho_k}, \mathfrak{Q})$$

and the character $\mathrm{Res}_{K_n}\, \chi_{\rho_k}$ factors through the extension $K_n(\sqrt[p^n]{\Delta})/K_n$. The prime $\mathfrak{Q}$ is totally yet tamely ramified in this extension. Therefore $\mathrm{Res}_{K_n}\, \chi_{\rho_k}$ is non-trivial on the inertia group, but is trivial on all the higher ramification groups. By definition of Artin conductor, this implies $f(\tilde{\chi}, \mathfrak{Q}) = 1$. Therefore,

$$\begin{aligned} \mathrm{ord}_q(N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})) &= \sum_{\mathfrak{Q}|q} [k_{n,\mathfrak{Q}} : \mathbb{F}_q] \\ &= [k_{n,\mathfrak{Q}} : \mathbb{F}_q] \times \text{number of primes of } K_n \text{ above } q \\ &= [K_n : \mathbb{Q}] \end{aligned}$$

as q is unramified in $K_n/\mathbb{Q}$. Observing that $[K_n : \mathbb{Q}] = p^n - p^{n-1}$ completes the demonstration of (ii).

Finally, as α_p was chosen to be a p-adic unit and either choice of α_q is a p-adic unit when $q \neq p$, it is clear $\alpha(\mathfrak{c}(\mathbf{g}_{\rho_k/F_n} \otimes \psi))$ is always a p-adic unit. □

The following result allows us to link the automorphic form $\Phi_\psi^{n,k}$ with Artin-twists of the Hasse-Weil L-function of $E_{/F_n}$.

THEOREM 3.2. — *Let* $\tilde{\chi} = \mathrm{Res}_{K_n}(\chi_{\rho_k} \otimes \psi)$, *and* $A_{\tilde{\chi}} = N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})$. *Then*

$$\frac{i^{h_{F_n}}(-4i)^{\phi(p^n)/2}}{\alpha(\mathfrak{m}'\mathfrak{l}')C(\mathfrak{c}(\mathbf{f}),\mathbf{f})} \quad \mathcal{L}_{F_n}\left(\Phi_{\psi}^{n,k}\middle| U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right)$$
$$= \frac{\epsilon_{F_n}(\rho_k/F_n \otimes \psi)}{\alpha_p^{f(\rho_k/F_n\otimes\psi,\mathfrak{p})}\prod_{q|\Delta}\alpha_q^{\mathrm{ord}_q(A_{\tilde{\chi}})}}$$
$$\times \prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n \otimes \psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]})}{P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha_{q_v}'^{-[f_{n,v}:\mathbb{F}_{q_v}]})}$$
$$\times \frac{L_S(1, E, \rho_k/F_n \otimes \psi^{-1})}{\Omega_{E/F_n}^{\mathrm{Aut}}}$$

where h_{F_n} *is the narrow class number of* F_n.

Proof. — By its very definition,

$$P_v(\rho_k/F_n \otimes \psi, X) = (1 - \psi^{\dagger}(v)\beta_n(v)X)(1 - \psi^{\dagger}(v)\beta_n'(v)X).$$

Since $\alpha(v)$, $\alpha'(v)$ are the roots of the polynomial $X^2 - C(v,\mathbf{f})X + N_{F_n/\mathbb{Q}}(v)$, clearly $\alpha'(v)N_{F_n/\mathbb{Q}}(v)^{-1} = \alpha(v)^{-1}$ and

$$P_v(\rho_k/F_n \otimes \psi, \alpha(v)^{-1})$$
$$= (1 - \psi^{\dagger}(v)\beta_n(v)\alpha'(v)N_{F_n/\mathbb{Q}}(v)^{-1})(1 - \psi^{\dagger}(v)\beta_n'(v)\alpha'(v)N_{F_n/\mathbb{Q}}(v)^{-1}).$$

Applying a similar formula for $P_v(\rho_k/F_n \otimes \psi, \alpha'(v)^{-1})$, we discover

$$\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1)$$
$$= \prod_{v|\mathfrak{p}\mathfrak{l}_0} P_v(\rho_k/F_n \otimes \psi, \alpha(v)^{-1})P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha(v)^{-1}).$$

Now the Euler factor of $\Psi(1, \mathbf{f}_{/F_n}, \mathbf{g}_{\rho_k/F_n} \otimes \psi)$ at the primes in S is

$$\prod_{v|\mathfrak{p}\mathfrak{l}_0} P_v(\rho_k/F_n \otimes \psi, \alpha(v)^{-1})^{-1} P_v(\rho_k/F_n \otimes \psi, \alpha'(v)^{-1})^{-1}$$

in which case

$$\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1) \cdot \Psi(1, \mathbf{f}_{/F_n}, \mathbf{g}_{\rho_k/F_n} \otimes \psi^{-1})$$
$$= \prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n \otimes \psi, \alpha(v))}{P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha'(v))} \cdot \Psi_S(1, \mathbf{f}_{/F_n}, \mathbf{g}_{\rho_k/F_n} \otimes \psi^{-1}).$$

We showed earlier that $\alpha(v) = \alpha_{q_v}^{[f_{n,v}:\mathbb{F}_q]}$ and $\alpha'(v) = \alpha_{q_v}'^{[f_{n,v}:\mathbb{F}_q]}$ where q_v denotes the unique rational prime below v. This gives us the required factor

$$\prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n \otimes \psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]})}{P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha_{q_v}'^{-[f_{n,v}:\mathbb{F}_{q_v}]})}.$$

Further, because E is semistable over $\mathbb{Q}$ we have an equality

$$\frac{\Psi_S(1, \mathbf{f}_{/F_n}, \mathbf{g}_{\rho_k/F_n \otimes \psi^{-1}})}{\left\langle \mathbf{f}_{/F_n}, \mathbf{f}_{/F_n} \right\rangle_{\mathfrak{c}(\mathbf{f})}} = \frac{L_S(1, E, \rho_k/F_n \otimes \psi^{-1})}{\Omega_{E/F_n}^{\mathrm{Aut}}}.$$

One concludes that

$$\begin{aligned}\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1) \times \frac{\Psi(1, \mathbf{f}_{/F_n}, \mathbf{g}_{\rho_k/F_n} \otimes \psi^{-1})}{\left\langle \mathbf{f}_{/F_n}, \mathbf{f}_{/F_n} \right\rangle_{\mathfrak{c}(\mathbf{f})}} \\ = \prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n \otimes \psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]})}{P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha_{q_v}'^{-[f_{n,v}:\mathbb{F}_{q_v}]})} \times \frac{L_S(1, E, \rho_k/F_n \otimes \psi^{-1})}{\Omega_{E/F_n}^{\mathrm{Aut}}}.\end{aligned}$$

Applying Corollary 2.11 and Lemma 3.1 to this, the result follows. □

3.2. The Kummer congruences

We will now prove that we have constructed a integral measure. Let $S = \mathrm{supp}(\mathfrak{p}\mathfrak{l}_0)$, and ψ be a character of $\mathcal{G}_n = \mathrm{Gal}(F_{n,S}^{\mathrm{ab}}/F_n)$. Consider the algebraic distribution on $\mathcal{G}_n$ given by

$$\int_{x \in \mathcal{G}_n} \psi(x) d\mu(x) := \frac{i^{h_{F_n}}(-4i)^{\phi(p^n)/2}}{\alpha(\mathfrak{m}'\mathfrak{l}')C(\mathfrak{c}(\mathbf{f}), \mathbf{f})} \quad \mathcal{L}_{F_n}\left(\Phi_\psi^{n,k} \middle| U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right)$$

where the Hilbert modular form

$$\begin{aligned}\Phi_\psi^{n,k} &= \Phi^{n,k}(\rho_k/F_n \otimes \psi, \mathfrak{c}(\mathbf{f}_{/F_n})\mathfrak{m}'\mathfrak{l}') \\ &= (\mathbf{g}_{\rho_k/F_n \otimes \psi}) \times E_1\left(0, \mathfrak{c}(\mathbf{f}_{/F_n})\mathfrak{m}'\mathfrak{l}', (\mathrm{Res}_{F_n} \det \rho_k)^{\dagger -1} \otimes \psi^{\dagger -2}\right)\end{aligned}$$

as in the previous section.

Proposition 3.3. — *The above distribution μ is a p-bounded measure on $\mathcal{G}_n$.*

Proof. — To show μ is bounded, it suffices to check the Kummer Congruences. In other words, if there exist $b_\psi \in \mathbb{C}_p$ (with only finitely many b_ψ non-zero) such that

$$\sum_\psi b_\psi \psi(x) \in p^m \mathcal{O}_{\mathbb{C}_p}$$

for all $x \in \mathcal{G}_n$, then

$$\sum_{\psi} Bb_\psi \int_{x\in\mathcal{G}_n} \psi(x)d\mu(x) \in p^m\mathcal{O}_{\mathbb{C}_p}$$

for some fixed constant $B \in \mathbb{Z}$.

From Atkin-Lehner theory, we know the linear functional $\mathcal{L}_{F_n}$ decomposes into a finite linear combination of the Fourier coefficients. So there exist finitely many ideals $\mathfrak{n}_i$ and fixed algebraic numbers $l(\mathfrak{n}_i) \in \overline{\mathbb{Q}}$ such that

$$\mathcal{L}_{F_n}(\Theta) = \sum_i C(\mathfrak{n}_i, \Theta) l(\mathfrak{n}_i)$$

for all $\Theta \in \mathcal{M}_2(\mathfrak{c}_0)$. Therefore, putting

$$u = \frac{i^{h_{F_n}}(-4i)^{\phi(p^n)/2}}{C(\mathfrak{c}(\mathbf{f}), \mathbf{f})\alpha(\mathfrak{m}'\mathfrak{l}')}$$

(which is a p-adic unit), we have

$$\begin{aligned}\sum_{\psi} Bb_\psi \int_{x\in\mathcal{G}_n} \psi(x)d\mu(x) &= uB\sum_{\psi} b_\psi \mathcal{L}_{F_n}\left(\Phi_\psi^{n,k}\middle|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1})\right)\\ &= uB\sum_{\psi} b_\psi \sum_i C(\mathfrak{n}_i, \Phi_\psi^{n,k}|U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}))l(\mathfrak{n}_i)\\ &= u\sum_i\left(\sum_{\psi} b_\psi C(\mathfrak{n}_i\mathfrak{p}\mathfrak{l}_0\mathfrak{m}'^{-1}\mathfrak{l}'^{-1}, \Phi_\psi^{n,k})\right)Bl(\mathfrak{n}_i).\end{aligned}$$

We now choose $B \in \mathbb{Z}$ so that $l(\mathfrak{n}_i)B \in \mathcal{O}_{\mathbb{C}_p}$ for all i. The above formula means it suffices to prove $\sum_\psi b_\psi C(\mathfrak{n}, \Phi_\psi^{n,k}) \in p^m\mathcal{O}_{\mathbb{C}_p}$ for any ideal $\mathfrak{n}$.

From Corollary 2.11, we know that the λ-component of $\Phi_\psi^{n,k}$ has Fourier coefficients

$$\begin{aligned}\phi_{\psi,\lambda}^{n,k}(\xi) = \sum_{\xi=\xi_1+\xi_2} \sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_K,\\ \mathfrak{a}\bar{\mathfrak{a}}=\xi_1\tilde{t}_\lambda^{-1}}} &(\chi_{\rho_k}^\dagger \circ N_{K_n/K_k})(\mathfrak{a})\psi^\dagger(\xi_1\tilde{t}_\lambda^{-1})\\ &\cdot \sum_{\substack{\tilde{\xi_2}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_{F_n},\\ b\in\tilde{t}_\lambda}} ((\det\rho_k)^\dagger \circ N_{F_n/F_k})^{-1}(\tilde{c})\psi^\dagger(\tilde{c})^{-2}.\end{aligned}$$

Recall that $C(\mathfrak{n}, \Phi_\psi^{n,k}) = N_{F_n/\mathbb{Q}}(\tilde{t}_\lambda)^{-1}\phi_{\psi,\lambda}^{n,k}(\xi)$ when $\mathfrak{n} = \xi\tilde{t}_\lambda^{-1}$. Therefore

$$\sum_\psi b_\psi C(\mathfrak{n}, \Phi_\psi^{n,k}) = N_{F_n/\mathbb{Q}}(\tilde{t}_\lambda)^{-1} \sum_\psi b_\psi \sum_{\xi_1,\xi_2} \sum_{\mathfrak{a}} (\chi_{\rho_k}^\dagger \circ N_{K_n/K_k})(\mathfrak{a}) \psi^\dagger(\xi_1 \tilde{t}_\lambda^{-1}) \cdot \sum_c ((\det \rho_k)^\dagger \circ N_{F_n/F_k})^{-1}(\tilde{c}) \psi^\dagger(\tilde{c})^{-2}$$
$$= N_{F_n/\mathbb{Q}}(\tilde{t}_\lambda)^{-1} \sum_{\xi_1,\xi_2} \sum_{\mathfrak{a}} (\chi_{\rho_k}^\dagger \circ N_{K_n/K_k})(\mathfrak{a}) \cdot \sum_c ((\det \rho_k)^\dagger \circ N_{F_n/F_k})^{-1}(\tilde{c}) \Bigg(\sum_\psi b_\psi \psi^\dagger(\xi_1 \tilde{t}_\lambda^{-1} \tilde{c}^{-2}) \Bigg).$$

By assumption, $\sum_\psi b_\psi \psi^\dagger(\xi_1 \tilde{t}_\lambda^{-1} \tilde{c}^{-2}) \in p^m \mathcal{O}_{\mathbb{C}_p}$, and t_λ is always chosen so that $N_{F_n/\mathbb{Q}}(\tilde{t}_\lambda)$ is prime to p. Therefore $\sum_\psi b_\psi C(\mathfrak{n}, \Phi_\psi^{n,k}) \in p^m \mathcal{O}_{\mathbb{C}_p}$. □

THEOREM 3.4. — *If $\mathbf{f}_{/F_n}$ is not congruent modulo $\mathfrak{p}$ to a distinct element of $\mathcal{M}_2(\mathfrak{c}(\mathbf{f}_{/F_n}))$, then there exists an abelian p-adic L-function $\mathbf{L}_{p,\Delta}(E, \rho_k/F_n, \boldsymbol{\alpha})$ in $\mathcal{O}_{\mathbb{C}_p}[[\mathcal{G}_n]]$ interpolating the special values*

$$\frac{\epsilon_{F_n}(\rho_k/F_n \otimes \psi)}{\alpha_p^{f(\rho_k/F_n \otimes \psi, \mathfrak{p})} \prod_{q|\Delta} \alpha_q^{\mathrm{ord}_q(A_{\tilde{\chi}})}} \times \prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n \otimes \psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]})}{P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha_{q_v}'^{-[f_{n,v}:\mathbb{F}_{q_v}]})} \times \frac{L_S(1, E, \rho_k/F_n \otimes \psi^{-1})}{\Omega_{E/F_n}^{\mathrm{Aut}}}$$

at all finite characters ψ of $\mathcal{G}_n = \mathrm{Gal}(F_{n,S}^{\mathrm{ab}}/F_n)$.

Here $\tilde{\chi} = \mathrm{Res}_{K_n}(\chi_{\rho_k} \otimes \psi)$, $A_{\tilde{\chi}} = N_{K_n/\mathbb{Q}}(\mathfrak{f}_{\tilde{\chi}})$, and $\boldsymbol{\alpha} = (\alpha_{q_1}, \ldots, \alpha_{q_r})$ denotes our choice of α_q for each prime $q|\Delta$.

Proof. — By Proposition 3.3, there exists an element

$$\mathbf{L}_{p,\Delta}(E, \rho_k/F_n, \boldsymbol{\alpha}) \in \mathcal{O}_{\mathbb{C}_p}[[\mathcal{G}_n]] \otimes_{\mathbb{Z}} \mathbb{Q}$$

which has special values $\int_{x \in \mathcal{G}_n} \psi(x) d\mu(x)$ at all characters ψ of $\mathcal{G}_n$. We will show that these special values are in fact p-integral. We have

$$\int_{x \in \mathcal{G}_n} \psi(x) d\mu(x) = \frac{i^{h_{F_n}} (-4i)^{\phi(p^n)/2}}{C(\mathfrak{c}(\mathbf{f}), \mathbf{f}) \alpha(\mathfrak{m}'\mathfrak{l}')} \quad \mathcal{L}_{F_n} \left(\Phi_\psi^{n,k} \middle| U(\mathfrak{m}'\mathfrak{l}'\mathfrak{p}^{-1}\mathfrak{l}_0^{-1}) \right)$$
$$= (\ p\text{-adic unit}\ ) \times \mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1) \times \epsilon_{F_n}(\rho_{k/F_n} \otimes \psi) \times \frac{\Psi(1, \mathbf{f}_{/F_n}, (\mathbf{g}_{\rho_k/F_n \otimes \psi})^\iota)}{\left\langle \mathbf{f}_{/F_n}, \mathbf{f}_{/F_n} \right\rangle_{\mathfrak{c}(\mathbf{f})}}$$

by Corollary 2.11. Recall that

$$\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1) = \prod_{v|\mathfrak{p}\mathfrak{l}_0} P_v(\rho_k/F_n \otimes \psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]}) P_v(\rho_k/F_n \otimes \psi^{-1}, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]}).$$

The polynomials $P_{\mathfrak{q}}(\rho_k/F_n \otimes \psi^{-1}, X)$ all have p-integral coefficients, and α_q is p-integral for any $q \neq p$, hence

$$P_{\mathfrak{q}}(\rho_k/F_n \otimes \psi, \alpha_q^{-[f_{n,\mathfrak{q}}:\mathbb{F}_q]}) \in \mathcal{O}_{\mathbb{C}_p}$$

when $\mathfrak{q} \neq \mathfrak{p}$. Also α_p was chosen to be the unit root, thus we also have

$$P_{\mathfrak{p}}(\rho_k/F_n \otimes \psi, \alpha_p^{-1}) \in \mathcal{O}_{\mathbb{C}_p}.$$

The same applies when ψ is replaced by ψ^{-1}, and $\mathrm{Eul}_{\mathfrak{p}\mathfrak{l}_0}(\rho_k/F_n \otimes \psi^{-1}, 1)$ is p-integral too. Lastly, using Theorem 2.4 with $\rho = (\rho_k/F_n) \otimes \psi$ yields

$$\epsilon_{F_n}(\rho_{k/F_n} \otimes \psi) \frac{\Psi(s, \mathbf{f}_{/F_n}, (\mathbf{g}_{\rho_k/F_n \otimes \psi})^{\iota})}{\left\langle \mathbf{f}_{/F_n}, \mathbf{f}_{/F_n} \right\rangle_{\mathfrak{c}(\mathbf{f})}} \in \mathcal{O}_{\mathbb{C}_p}.$$

All special values therefore lie in $\mathcal{O}_{\mathbb{C}_p}$, so $\mathbf{L}_{p,\Delta}(E, \rho_k/F_n, \boldsymbol{\alpha}) \in \mathcal{O}_{\mathbb{C}_p}[[\mathcal{G}_n]]$. $\square$

3.3. The weak form of Kato's congruences

In this section we will use our distributions to prove a final set of congruences. We do this as evidence for the stronger congruences from Kato's paper [9], which imply the existence of a non-abelian p-adic L-function. Firstly, define $(a_n)_{n \geqslant 0} \in \prod_{n \geqslant 0} \mathbb{Z}_p[[U^{(n)}]]^{\times}$ by

$$a_n = \mathbf{L}_p(E, \rho_n/F_n)$$

for each $n \geqslant 0$, and recall that $b_n = a_n/N_{0,n}(a_0)$ and $c_n = b_n/\phi(b_{n-1})$. Ideally we want to prove that

$$\prod_{1 \leqslant i \leqslant n} N_{i,n}(c_i)^{p^i} \equiv 1 \mod p^{2n}$$

for all $n \geqslant 1$. We are unable to prove this congruence modulo p^{2n}, but we will at least prove it modulo p^{n+1}.

Lemma 3.5. — *For all $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$,*

$$a_n \equiv N_{i,n}(a_i) \mod p\mathbb{Z}_p[[U^{(n)}]].$$

Proof. — Firstly, we observe that the element $N_{i,n}(a_i)$ has special values

$$\psi(N_{i,n}(a_i)) = \frac{\epsilon_{F_n}(\rho_i/F_n\otimes\psi)_{\mathfrak{p}}}{\alpha_p^{f(\rho_i/F_n\otimes\psi,\mathfrak{p})}} \times \frac{P_{\mathfrak{p}}(\rho_i/F_n\otimes\psi, \alpha_p^{-[F_n:\mathbb{Q}]})}{P_{\mathfrak{p}}(\rho_i/F_n\otimes\psi^{-1}, \alpha_p'^{-[F_n:\mathbb{Q}]})} \times \frac{L_S(1,E,\rho_i/F_n\otimes\psi)}{(\Omega_E^+\Omega_E^-)^{\phi(p^n)/2}}$$

at all finite characters ψ of $U^{(n)}$. Let's abuse notation slightly, and write $\mathbf{L}_{p,\Delta}(E,\rho_i/F_n,\boldsymbol{\alpha})$ for the image of $\mathbf{L}_{p,\Delta}(E,\rho_i/F_n,\boldsymbol{\alpha})$ under the projection

$$\mathcal{O}_{\mathbb{C}_p}[[\mathcal{G}_n]] \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]].$$

Remark. — The representation $\rho_k/F_n\otimes\psi$ factors through M/F_n for some field M that depends on ψ. By definition $P_v(\rho_k/F_n\otimes\psi,X)$ is the characteristic polynomial of Frob_v acting on the inertia invariant subspace $(\rho_k/F_n\otimes\psi)^{I_{\bar v}}$. Here $\bar v$ is a place of M dividing v and $I_{\bar v}\subseteq \mathrm{Gal}(M/F_n)_{\bar v}$ is the inertia subgroup. However, since we are now considering characters ψ which are ramified only above p, if $v|\Delta$ then $\mathrm{Res}_{I_{\bar v}}\,\psi = \mathbf{1}$. As a corollary

$$(\rho_k/F_n\otimes\psi)^{I_{\bar v}} = (\rho_k/F_n)^{I_{\bar v}}.$$

Because ρ_k/F_n is induced from a character over K_n, and v is unramified in K_n/F_n, one easily checks that $\mathrm{Res}_{I_{\bar v}}\,\rho_k/F_n$ decomposes as a sum of two characters. This implies $(\rho_k/F_n)^{I_{\bar v}} = 0$. Thus we have shown $P_v(\rho_k/F_n\otimes\psi,X) = 1$ for each v dividing Δ, which permits the simplification

$$\prod_{v|\mathfrak{p}\mathfrak{l}_0} \frac{P_v(\rho_k/F_n\otimes\psi, \alpha_{q_v}^{-[f_{n,v}:\mathbb{F}_{q_v}]})}{P_v(\rho_k/F_n\otimes\psi^{-1}, \alpha_{q_v}'^{-[f_{n,v}:\mathbb{F}_{q_v}]})} = \frac{P_{\mathfrak{p}}(\rho_k/F_n\otimes\psi, \alpha_p^{-[F_n:\mathbb{Q}]})}{P_{\mathfrak{p}}(\rho_k/F_n\otimes\psi^{-1}, \alpha_p'^{-[F_n:\mathbb{Q}]})}.$$

Comparing the special values of $N_{i,n}(a_i)$ and $\mathbf{L}_{p,\Delta}(E,\rho_i/F_n,\boldsymbol{\alpha})$, one finds

$$N_{i,n}(a_i) = \frac{\Omega_{E/F_n}^{\mathrm{Aut}}}{(\Omega_E^+\Omega_E^-)^{\phi(p^n)/2}} \times \gamma_E^{i,n} \times \mathbf{L}_{p,\Delta}(E,\rho_i/F_n,\boldsymbol{\alpha})$$

where $\gamma_E^{i,n}\in\mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$ satisfies

$$\psi(\gamma_E^{i,n}) = \frac{\prod_{q|\Delta}\alpha_q^{p^n-p^{n-1}}}{\prod_{v\neq\mathfrak{p}}\epsilon_{F_n}(\rho_k/F_n\otimes\psi^{-1})_v}.$$

Let $\mathfrak{M}_{\mathbb{C}_p}$ denote the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$.

Claim ($\star$). — $\gamma_E^{0,n}\equiv\gamma_E^{1,n}\equiv\cdots\equiv\gamma_E^{n,n} \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]]$ *for each* $n\in\mathbb{N}$.

We will prove this claim at the end of this section (*cf.* Lemma 3.8).

To make further progress, we recall an important conjecture made by Doi, Hida and Ishii. This is Conjecture 1.3 of [4], and we state it here in a simplified form. Let A be a commutative ring, and

$$\lambda : h_\kappa(\Gamma_0(N), \psi; A) \longrightarrow \mathbb{C}$$

a specialisation of the Hecke algebra, corresponding to a primitive form $f \in S_\kappa(\Gamma_0(N), \psi; A)$. Moreover, let $\hat{\lambda}$ denote the base-change lift of λ to the totally real field F. Then in the notation of [4]:

Conjecture 3.6 (Doi-Hida-Ishii). — *Assume the Galois representation associated to λ is residually absolutely irreducible, its characters are distinguished at $\mathfrak{p}$, and that $\lambda(T(p)) \in A^\times$. Then for each character $\theta : S_F \to \{\pm 1\}$,*

$$\Omega(\theta, \hat{\lambda}; A) = \prod_{\sigma \in \Sigma_F} \Omega(\theta(-1_\sigma), \lambda; A)$$

up to A-units.

In the sequel, we assume their prediction holds for $A = \mathbb{Z}_p$, $f = f_E$ and $F = F_n$. In this case $\Omega(\pm, \lambda; A) = \Omega_E^\pm$, the Néron periods associated to E. In fact up to a sign $\Omega(\theta, \hat{\lambda}; A) = \Omega^{\mathrm{Aut}}_{\mathbf{f}/F_n}$, the automorphic period we defined earlier. Therefore their conjecture implies

$$\begin{aligned} \mathbb{Z}_p \cdot \Omega^{\mathrm{Aut}}_{\mathbf{f}/F_n} &\overset{\text{D-H-I}}{=} \mathbb{Z}_p \cdot \left(\Omega^+_{f_E/\mathbb{Q}} \Omega^-_{f_E/\mathbb{Q}}\right)^{[F_n:\mathbb{Q}]} \\ &= \mathbb{Z}_p \cdot \left(\Omega_E^+ \Omega_E^-\right)^{(p^n - p^{n-1})/2}. \end{aligned}$$

In particular, $\Omega^{\mathrm{Aut}}_{E/F_n} \times (\Omega_E^+ \Omega_E^-)^{-\phi(p^n)/2}$ is a p-adic unit. Alternatively, one can bypass this conjecture by instead assuming that all ρ_n-twists of the homology of $X_1(N_E)$ are p-integral, in the sense of Stevens [13].

Step 1. — We will first show that

$$\mathbf{L}_{p,\Delta}(E, \rho_i/F_n, \boldsymbol{\alpha}) \equiv \mathbf{L}_{p,\Delta}(E, \rho_n, \boldsymbol{\alpha}) \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]].$$

This is the same as showing

$$\int_{U^{(n)}} \psi(g) d\mu_{\rho_i/F_n}(g) \equiv \int_{U^{(n)}} \psi(g) d\mu_{\rho_n}(g) \mod \mathfrak{M}_{\mathbb{C}_p}$$

for all characters ψ of $U^{(n)}$. We have already proved these measures are integral. Following the argument from the proof of Proposition 3.3, it suffices to show

$$C(\mathfrak{m}, \Phi_\psi^{i,n}) \equiv C(\mathfrak{m}, \Phi_\psi^{n,n}) \mod \mathfrak{M}_{\mathbb{C}_p}$$

for all $\mathfrak{m}$ and ψ. Fix an ideal $\mathfrak{m} = \xi\tilde{t}_\lambda^{-1}$. Let us use the fact that the characters χ_{ρ_i} satisfy

$$\chi_{\rho_i} \equiv \mathbf{1} \mod \mathfrak{M}_{\mathbb{C}_p}$$

for each $k \geqslant 0$. Further,

$$(\det \rho_i)^\dagger(\mathfrak{b}) = \theta_{K_i/F_i}(\mathfrak{b})\chi_{\rho_i}^\dagger(\mathfrak{b}\mathcal{O}_{K_i})$$

hence

$$(\det \rho_i)^\dagger(\mathfrak{b}) \equiv \theta_{K_i/F_i}(\mathfrak{b}) \mod \mathfrak{M}_{\mathbb{C}_p}$$

for each k. Therefore, we have congruences

$$\begin{aligned} C(\mathfrak{m}, \Phi_\psi^{i,n}) &= \sum_{\xi_1+\xi_2=\xi} \sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_{K_n},\\ \mathfrak{a}\bar{\mathfrak{a}}=\xi_1\tilde{t}_\lambda^{-1}}} (\chi_{\rho_i}^\dagger \circ N_{K_n/K_i})(\mathfrak{a})\psi^\dagger(\mathfrak{a}\bar{\mathfrak{a}}) \\ &\qquad \cdot \sum_{\substack{\tilde{\xi_2}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_{F_n},\\ b\in\tilde{t}_\lambda}} ((\det \rho_i)^\dagger \circ N_{F_n/F_i})^{-1}(\tilde{c})\psi^\dagger(\tilde{c})^{-2} \\ &\equiv \sum_{\xi_1,\xi_2}\sum_{\mathfrak{a}} \psi^\dagger(\mathfrak{a}\bar{\mathfrak{a}}) \sum_c (\theta_{K_i/F_i} \circ N_{F_n/F_i})(\tilde{c})\psi^\dagger(\tilde{c})^{-2} \mod \mathfrak{M}_{\mathbb{C}_p}, \end{aligned}$$

and also

$$\begin{aligned} C(\mathfrak{m}, \Phi_\psi^{n,n}) &= \sum_{\xi_1+\xi_2=\xi} \sum_{\substack{\mathfrak{a}\lhd\mathcal{O}_{K_n},\\ \mathfrak{a}\bar{\mathfrak{a}}=\xi_1\tilde{t}_\lambda^{-1}}} \chi_{\rho_n}^\dagger(\mathfrak{a})\psi^\dagger(\mathfrak{a}\bar{\mathfrak{a}}) \sum_{\substack{\tilde{\xi_2}=\tilde{b}\tilde{c},\\ c\in\mathcal{O}_{F_n},\\ b\in\tilde{t}_\lambda}} (\det \rho_i)^{\dagger-1}(\tilde{c})\psi^\dagger(\tilde{c})^{-2} \\ &\equiv \sum_{\xi_1,\xi_2}\sum_{\mathfrak{a}} \psi^\dagger(\mathfrak{a}\bar{\mathfrak{a}}) \sum_c \theta_{K_n/F_n}(\tilde{c})\psi^\dagger(\tilde{c})^{-2} \mod \mathfrak{M}_{\mathbb{C}_p}. \end{aligned}$$

We will be done if we can show

$$\theta_{K_i/F_i} \circ N_{F_n/F_i} = \theta_{K_n/F_n}.$$

Recall that θ_{K_i/F_i} is given on primes by

$$\theta_{K_i/F_i}(\mathfrak{q}) = \begin{cases} 1 & \text{if } \mathfrak{q} \text{ splits in } K_i/F_i \\ -1 & \text{if } \mathfrak{q} \text{ is inert in } K_i/F_i \\ 0 & \text{if } \mathfrak{q} \text{ ramifies in } K_i/F_i \end{cases}.$$

The only prime which ramifies in K_n/F_n is $\mathfrak{p}$, and clearly

$$\theta_{K_i/F_i} \circ N_{F_n/F_i}(\mathfrak{p}) = \theta_{K_n/F_n}(\mathfrak{p}) = 0.$$

It remains to check this for the non-ramified primes. Let $\mathfrak{Q}$ and $\mathfrak{q}$ be primes of F_n and F_i respectively, both coprime to p and such that $\mathfrak{Q}|\mathfrak{q}$. Let $f = [f_{n,\mathfrak{Q}} : f_{i,\mathfrak{q}}]$, so that $N_{F_n/F_i}(\mathfrak{Q}) = \mathfrak{q}^f$. Then we must check

$$\theta_{K_i/F_i}(\mathfrak{q})^f = \theta_{K_n/F_n}(\mathfrak{Q}).$$

Since $[K_n : F_n] = [K_i : F_i] = 2$ and $[F_n : F_i] = p^{n-i}$, $\mathfrak{Q}$ splits in K_n/F_n if and only if $\mathfrak{q}$ splits in K_i/F_i. Therefore,

$$\theta_{K_i/F_i}(\mathfrak{q}) = \theta_{K_n/F_n}(\mathfrak{Q}).$$

Also f divides $[F_n : F_i] = p^{n-i}$ which means f is odd. As a consequence

$$\theta_{K_i/F_i}(\mathfrak{q})^f = \theta_{K_i/F_i}(\mathfrak{q}) = \theta_{K_n/F_n}(\mathfrak{Q}).$$

We have verified $\theta_{K_i/F_i} \circ N_{F_n/F_i} = \theta_{K_n/F_n}$, which completes Step 1.

Step 2. — The congruence

$$\gamma_E^{i,n} \equiv \gamma_E^{n,n} \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]]$$

follows from Claim $(\star)$, and the congruence

$$\mathbf{L}_{p,\Delta}(E, \rho_i/F_n, \boldsymbol{\alpha}) \equiv \mathbf{L}_{p,\Delta}(E, \rho_n/F_n, \boldsymbol{\alpha}) \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]]$$

from Step 1.

Putting these together, then multiplying by $\Omega^{\mathrm{Aut}}_{E/F_n}(\Omega^+_E\Omega^-_E)^{-\phi(p^n)/2}$ (which is a p-adic unit), we obtain

$$\frac{\Omega^{\mathrm{Aut}}_{E/F_n}}{(\Omega^+_E\Omega^-_E)^{\phi(p^n)/2}}\gamma_E^{i,n}\mathbf{L}_{p,\Delta}(E, \rho_i/F_n, \boldsymbol{\alpha}) \equiv \frac{\Omega^{\mathrm{Aut}}_{E/F_n}}{(\Omega^+_E\Omega^-_E)^{\phi(p^n)/2}}\gamma_E^{n,n}\mathbf{L}_{p,\Delta}(E, \rho_n/F_n, \boldsymbol{\alpha}) \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]],$$

i.e.

$$N_{i,n}(\mathbf{L}_p(E, \rho_i)) \equiv \mathbf{L}_p(E, \rho_n) \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]].$$

Step 3. — We now quote Theorem 4.2 of [2], which states that

$$\epsilon_{F_n}(\rho)_{\mathfrak{p}}\frac{L_S(1, E, \rho)}{(\Omega^+_E)^{\dim\rho^+}(\Omega^-_E)^{\dim\rho^-}} \in \mathbb{Q}(\rho)$$

where $\mathbb{Q}(\rho)$ denotes the field of definition of the Artin representation ρ. However as Dokchitser comments in [6], we actually have $\mathbb{Q}(\rho_i) = \mathbb{Q}$. Therefore

$$\mathbf{L}_p(E, \rho_i) \in \mathbb{Z}_p[[U^{(i)}]] \otimes \mathbb{Q} \quad \text{for all } i.$$

Further, our integrality result (Theorem 2.4) implies that the special values of

$$\mathbf{L}_p(E, \rho_i) = \frac{\Omega^{\mathrm{Aut}}_{E/F_i}}{(\Omega^+_E\Omega^-_E)^{\phi(p^i)/2}} \times \gamma_E^{i,i} \times \mathbf{L}_{p,\Delta}(E, \rho_i, \boldsymbol{\alpha})$$

are all p-integral; we deduce that

$$\mathbf{L}_p(E, \rho_i) \in \mathbb{Z}_p[[U^{(i)}]] \quad \text{for all } i.$$

In conclusion

$$N_{i,n}(\mathbf{L}_p(E,\rho_i)) \equiv \mathbf{L}_p(E,\rho_n) \mod \left(\mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]] \cap \mathbb{Z}_p[[U^{(n)}]]\right),$$

i.e.

$$N_{i,n}(\mathbf{L}_p(E,\rho_i)) \equiv \mathbf{L}_p(E,\rho_n) \mod p\mathbb{Z}_p[[U^{(n)}]]$$

which is the desired congruence. □

We can now prove the main result of this section.

Theorem 3.7.

$$\prod_{1\leqslant i\leqslant n} N_{i,n}(c_i)^{p^i} \equiv 1 \mod p^{n+1}$$

for all $n \geqslant 1$.

Proof. — From the definitions of c_i and b_i, this congruence can be rearranged into the form

$$\prod_{1\leqslant i\leqslant n} N_{i,n}(a_i\cdot\phi\circ N_{0,i-1}(a_0))^{p^i} \equiv \prod_{1\leqslant i\leqslant n} N_{i,n}(\phi(a_{i-1})\cdot N_{0,i}(a_0))^{p^i} \mod p^{n+1}.$$

We prove this result by induction on n.

Base Case n=1. — We need to prove

$$(\phi(a_0)\cdot a_1)^p \equiv (\phi(a_0)\cdot N_{0,1}(a_0))^p \mod p^2.$$

First note that $x \equiv y \mod p$ implies $x^p \equiv y^p \mod p^2$, so it suffices to show

$$a_1 \equiv N_{0,1}(a_0) \mod p$$

which is a consequence of Lemma 3.5 (this also proves Theorem 1.3).

Induction Step. — Our inductive hypothesis is

$$\prod_{1\leqslant i\leqslant n-1} N_{i,n-1}(a_i\cdot\phi\circ N_{0,i-1}(a_0))^{p^i} \equiv \prod_{1\leqslant i\leqslant n-1} N_{i,n-1}(\phi(a_{i-1})\cdot N_{0,i}(a_0))^{p^i} \mod p^n.$$

Note that if $x_n \equiv y_n \mod p^r$ for $x_n, y_n \in \mathbb{Z}_p[[U^{(n)}]]$, then

$$N_{n-1,n}(x_n) \equiv N_{n-1,n}(y_n) \mod p^{r+1}.$$

Therefore our inductive hypothesis implies

$$N_{n-1,n}\left(\prod_{1\leqslant i\leqslant n-1} N_{i,n-1}(a_i\cdot\phi\circ N_{0,i-1}(a_0))^{p^i}\right) \equiv N_{n-1,n}\left(\prod_{1\leqslant i\leqslant n-1} N_{i,n-1}(\phi(a_{i-1})\cdot N_{0,i}(a_0))^{p^i}\right) \mod p^{n+1}$$

which can be rewritten as

$$(\dagger)\qquad \prod_{1\leqslant i\leqslant n-1} N_{i,n}(a_i\cdot\phi\circ N_{0,i-1}(a_0))^{p^i} \equiv \prod_{1\leqslant i\leqslant n-1} N_{i,n}(\phi(a_{i-1})\cdot N_{0,i}(a_0))^{p^i} \mod p^{n+1}.$$

Now, from Lemma 3.5 we know

$$a_{n-1}\equiv N_{0,n-1}(a_0) \mod p$$

implying that

$$\phi(a_{n-1})\equiv \phi(N_{0,n-1}(a_0)) \mod p.$$

Combining this with the congruence

$$a_n\equiv N_{0,n}(a_0) \mod p$$

(which also comes from Lemma 3.5), we obtain

$$N_{0,n}(a_0)\phi(a_{n-1})\equiv a_n\phi(N_{0,n-1}(a_0)) \mod p.$$

Finally, raising both sides to the p^n-th power yields

$$(N_{0,n}(a_0)\phi(a_{n-1}))^{p^n}\equiv (a_n\phi(N_{0,n-1}(a_0)))^{p^n} \mod p^{n+1}.$$

This provides the factor at $i=n$; multiplying by the congruence (†) above

$$\prod_{1\leqslant i\leqslant n} N_{i,n}(a_i\cdot\phi\circ N_{0,i-1}(a_0))^{p^i} \equiv \prod_{1\leqslant i\leqslant n} N_{i,n}(\phi(a_{i-1})\cdot N_{0,i}(a_0))^{p^i} \mod p^{n+1}$$

which completes the induction step.

□

It remains to prove Claim (⋆).

Lemma 3.8. — *For each $n\geqslant 0$,*

$$\gamma_E^{0,n}\equiv\gamma_E^{1,n}\equiv\cdots\equiv\gamma_E^{n,n} \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]]$$

where $\gamma_E^{k,n}\in\mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$ takes special values

$$\psi(\gamma_E^{k,n})=\frac{\prod_{q|\Delta}\alpha_q^{p^n-p^{n-1}}}{\prod_{v\neq\mathfrak{p}}\epsilon_{F_n}(\rho_k/F_n\otimes\psi^{-1})_v}.$$

Proof. — We will show that the special values of these elements are congruent modulo $\mathfrak{M}_{\mathbb{C}_p}$. Let $M_n=K_n(\sqrt[p^n]{\Delta})$, so ρ_k/F_n factors through $G=\mathrm{Gal}(M_n/F_n)$. We need to verify

$$\epsilon_{F_n}(\rho_k/F_n\otimes\psi)_v\equiv\epsilon_{F_n}(\rho_n\otimes\psi)_v \mod \mathfrak{M}_{\mathbb{C}_p}$$

for all places v of F_n such that $v|\Delta$.

Case 1: v splits in K_n/F_n. — Let $\bar{v}$ be a place of M_n above v; in this case, the decomposition group $G_{\bar{v}}$ is contained in $\mathrm{Gal}(M_n/K_n)$. Therefore,

$$\mathrm{Res}_{G_{\bar{v}}}(\rho_k/F_n \otimes \psi) \cong (\mathrm{Res}_{G_{\bar{v}}} \chi_{\rho_k} \otimes \psi) \oplus (\mathrm{Res}_{G_{\bar{v}}} \chi_{\rho_k} \otimes \psi)^{-1}.$$

Thus it suffices to check that the epsilon-factors of the characters themselves are congruent.

Case 2: v is inert in K_n/F_n. — In this case we apply the inductivity of local epsilon factors in degree zero (see [14] (3.4.8)). This gives us

$$\frac{\epsilon_{F_n}(\rho_k/F_n \otimes \psi)_v}{\epsilon_{F_n}(\mathbf{1} \oplus \eta)_v} = \frac{\epsilon_{K_n}(\mathrm{Res}_{K_n} \chi_{\rho_k} \otimes \psi)_w}{\epsilon_{K_n}(\mathbf{1})_w}$$

where w is a prime of K_n above v, and η is the quadratic character of K_n/F_n, so in fact $\mathrm{Ind}_{K_n}^{F_n} \mathbf{1} = \mathbf{1} \oplus \eta$. It can be shown that both $\epsilon_{F_n}(\mathbf{1} \oplus \eta)_v$ and $\epsilon_{K_n}(\mathbf{1})_w$ are p-adic units. Therefore we may write

$$\begin{aligned} \epsilon_{F_n}(\rho_k/F_n \otimes \psi)_v &= \frac{\epsilon_{F_n}(\mathbf{1} \oplus \eta)_v}{\epsilon_{K_n}(\mathbf{1})_w} \times \epsilon_{K_n}(\mathrm{Res}_{K_n} \chi_{\rho_k} \otimes \psi)_w \\ &= (p\text{-adic unit}) \times \epsilon_{K_n}(\mathrm{Res}_{K_n} \chi_{\rho_k} \otimes \psi)_w. \end{aligned}$$

Thus we are again reduced to proving the congruence for the epsilon factors of the characters.

In both cases, it is enough to prove for each place $w|q$ that

$$\epsilon_{K_n}(\chi)_w \equiv \epsilon_{K_n}(\chi')_w \mod \mathfrak{M}_{\mathbb{C}_p}$$

where χ and χ' are two characters over K_n, both tamely ramified at w and satisfying $\chi \equiv \chi' \mod \mathfrak{M}_{\mathbb{C}_p}$. Recall that these local ϵ-factors depend on a choice of Haar measure dx and a choice of additive character $\tau : (K_{n,w}, +) \to \mathbb{C}^\times$. Then $\epsilon_{K_n}(\chi)_w = \epsilon(\chi, \tau, dx)$ in the notation of Tate from [14], and we have the Gauss sum expression

$$\epsilon(\chi, \tau, dx) = \chi(\pi^{a(\chi)+n(\tau)})q^{n(\tau)-\delta/2} \sum_{u \in \mathcal{O}_w^\times \bmod \pi^{a(\chi)}} \chi(u)\tau\left(\frac{u}{\pi^{a(\chi)+n(\tau)}}\right)$$

and similarly

$$\epsilon(\chi', \tau, dx) = \chi'(\pi^{a(\chi')+n(\tau)})q^{n(\tau)-\delta/2} \sum_{u \in \mathcal{O}_w^\times \bmod \pi^{a(\chi')}} \chi(u)\tau\left(\frac{u}{\pi^{a(\chi')+n(\tau)}}\right).$$

Here, π is a uniformiser for $K_{n,w}$, q is the number of elements in the residue field of w, δ is the exponent of w in the different of K_n, $a(\chi)$ is the exponent of w in the conductor of χ, and $n(\tau)$ is an integer depending on the additive character τ.

Since we assumed that χ and χ' are both tamely ramified at w, we have $a(\chi) = a(\chi')$. Therefore, since we also assume that χ and χ' are congruent modulo $\mathfrak{M}_{\mathbb{C}_p}$, the two sums are congruent term-by-term. □

A short example. — Consider the semistable elliptic curve

$$E : y^2 + xy + y = x^3 - x^2 - x - 14$$

which is labelled 17A1 in Cremona's tables; it possesses good ordinary reduction at $p = 7$. Its 7-division polynomial can be shown (using MAGMA) to be irreducible over the field $\mathbb{Q}(\mu_7)^+$, which implies that $\rho_{E,7}\big|_{G_{\mathbb{Q}(\mu_7)^+}}$ is a residually irreducible Galois representation. As a corollary, there are no congruences modulo the prime above 7 between $\mathbf{f}_{/F_1}$ and any other modular form of the same level.

Moreover from [5], Table 7-17A1, we also know that $\mathbf{1}(a_0) = 5.7^0 + 2.7^1 + 2.7^2 + \dots$ which implies that $a_0 \in \mathbb{Z}_7[[U^{(0)}]]^\times$. The system of congruences $a_n \equiv N_{0,n}(a_0) \mod 7$ confirms $a_n \in \mathbb{Z}_7[[U^{(n)}]]^\times$, *provided* the conjecture of Doi-Hida-Ishii holds for all $n \geqslant 1$. Granted this is the case, by Theorem 1.2 one obtains

$$\prod_{1 \leqslant i \leqslant n} N_{i,n}(c_i)^{7^i} \equiv 1 \mod 7^{n+1}$$

for all such n.

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