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CONTINUOUS DIVISION OF DIFFERENTIAL OPERATORS

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Introduction.

The classical Weierstrass Division Theorem for convergent power series was extended by Hironaka and Grauert to the case of division by several series. Galligo put these extensions on firm ground by providing a systematic treatment and proving the continuous dependence of quotients and remainder of the division on the given data. The formal case being straightforward, the proof of convergence and continuity requires to work with suitable polycylinders and to establish intricate norm estimates for the involved series. The general idea that a power series can be interpreted as an arbitrarily small perturbation of its initial monomial is disguised by the technicalities of the proof. This defect was overcome by Hauser and Müller [HM] by approximating instead of the series directly the $\mathcal{O}_n$-linear map between power series spaces $\mathcal{O}_n^p$ defined by the given series. A short and direct proof became thus available. This quite general strategy is valid for different situations and can be applied equally in the polynomial context or for solutions of certain differential equations [HRT], [H]. The only ingredient from analysis is the convergence of the geometric series of a linear map between Banach spaces, provided this map has norm < 1. As such the argument is, though comparable in spirit, much less difficult than

the proof of the Constant Rank Theorem for analytic maps between power series spaces or the Nash-Moser Implicit Function Theorem.

Briançon-Maisonobe in one variable and Castro in general first proved division theorems for differential operators [BM], [C]. Full details appear in Castro's thesis but have not been published. Recently, Mebkhout and Narváez [MN] considered a canonical topology on the rings of linear differential operators with analytic coefficients and proved the continuity of division. Their ad-hoc proof is rather technical and long. They give lots of applications and show how instrumental continuous division of differential operators is in the theory of D-modules. In particular, they obtain a short proof of the faithfully flatness of the ring of infinite order differential operators over the ring of finite order differential operators.

The method of proof of [HM] is taken up in the present paper to establish the continuity of division by differential operators with analytic coefficients in a simple and transparent way. Due to the non-commutativity the same routine requires additional efforts to get the correct norm estimates. Actually, the pseudo-norms on the ring of differential operators have to be chosen with particular care in order to make the arguments work. Slight deviations from the setting immediately cause serious obstructions.

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1. Topological structure on the ring of differential operators.

Let $\mathcal{D}_{\mathbb{C}^n}$ be the sheaf of linear differential operators with coefficients in $\mathcal{O}_{\mathbb{C}^n}$, the sheaf of holomorphic functions on $\mathbb{C}^n$. Denote by $\mathcal{O}_n$ and by $\mathcal{D}_n$ the stalk at the origin of the sheaves $\mathcal{O}_{\mathbb{C}^n}$ and $\mathcal{D}_{\mathbb{C}^n}$ respectively.

In [MN], §2, Mebkhout and Narváez have introduced a natural metrizable locally convex structure on $\mathcal{D}_{\mathbb{C}^n}$. For instance, if $U \subset \mathbb{C}^n$ is an open poly-cylinder of polyradius $\sigma = (\sigma_i)$ and $E = \sum_\beta a_\beta D^\beta = \sum_{\alpha\beta} a_{\alpha\beta} x^\alpha D^\beta$ is a differential operator on U, where $x = (x_1, \ldots, x_n)$, $D = (D_1, \ldots, D_n)$ is the vector of partial derivatives $D_i = \partial_i$ and the $a_\beta = \sum_\alpha a_{\alpha\beta} x^\alpha$ are holomorphic functions on U, the topology of $\mathcal{D}_{\mathbb{C}^n}(U)$

is determined by the norms

$$|E|_\rho^L = \sum_{d=0}^{\infty} d! \left(\sum_{|\beta|=d} |a_\beta|_\rho L^\beta \right) = \sum_{\alpha\beta} |\beta|! |a_{\alpha\beta}| \rho^\alpha L^\beta,$$

where ρ is a polyradius with $\rho_i < \sigma_i$ and $L = (L_1, \ldots, L_n) > 0$. It turns out that this topology does not depend on the local coordinates and that the completion of $\mathcal{D}_{\mathbb{C}^n}(U)$ is the space of infinite order differential operators on U, see *loc. cit.*

In this paper we shall consider another way of indexing the norms above. For $s, t > 0$, weights $\lambda, \mu \in \mathbb{Q}_+^n$ and $E = \sum_{\alpha\beta} a_{\alpha\beta} x^\alpha D^\beta \in \mathcal{D}_n$ we set

$$|E|_{st} = \sum_{\alpha\beta} |a_{\alpha\beta}| \cdot |\beta|! \cdot s^{\lambda\alpha} \cdot t^{-\mu\beta},$$

where we omit any reference to the weights, often fixed, for simplicity. We have obviously $|E|_{st} = |E|_{s^\lambda}^{t^{-\mu}}$.

The factorial of $|\beta|$ is crucial, as in [MN], and appears also in the present proof. For λ fixed, denote by $\mathcal{D}_n(s)$ the subspace of $\mathcal{D}_n$ of operators of finite norm w.r.t. s (and w.r.t. some or any t, μ), filtered by the Banach spaces $\mathcal{D}_n(s)_d$ of operators of order $\leqslant d$. The induced topology on $\mathcal{D}_n$ does not depend on the choice of λ and μ.

For $U \subset \mathbb{C}^n$ an open poly-cylinder of polyradius $\sigma = s_0^\lambda$ (λ fixed), the topology of $\mathcal{D}_{\mathbb{C}^n}(U)$ is determined by any family of norms $|-|_{st}$, with $0 < s < s_0$ and $t^{-\mu} \gg 0$ w.r.t. the componentwise order (for instance, $0 < t < t_0$ and μ fixed, or $t \in]0, 1[$ fixed and all $\mu \gg 0$).

2. Continuity of division.

Any monomial order in $\mathbb{N}^{2n}$ allows to associate to a differential operator its initial monomial as the monomial of minimal exponent occurring in its symbol [C]. For finitely many given differential operators M_i there exists $(\lambda, \mu) \in \mathbb{N}_+^{2n}$ such that for each i there is a unique (α_i, β_i) in the support of M_i with $\lambda\alpha_i - \mu\beta_i$ minimal and such that $x^{\alpha_i} D^{\beta_i}$ is the initial monomial of M_i w.r.t. the chosen monomial order, see the addendum for a proof. In fact, taking weights $(\sigma, \rho) \in \mathbb{N}^{2n}$ which induce the monomial order on a sufficiently large subset of $\mathbb{N}^{2n}$ one may choose $\lambda = \sigma$ and $\mu = \mathbf{k} - \rho$ where $\mathbf{k}$ denotes the vector $(k, \ldots, k)$ in $\mathbb{N}^n$ for some large k.

Write $M_i = x^{\alpha_i} D^{\beta_i} + \sum_{\alpha\beta} c^i_{\alpha\beta} x^\alpha D^\beta = x^{\alpha_i} D^{\beta_i} + M'_i$. Then $\lambda(\alpha_i - \alpha) < \mu(\beta_i - \beta)$ for all (α, β) in the support of M_i.

Let $T = (T_1, \ldots, T_n)$ be the shift operator given by T_i $x^\alpha D^\beta = x^\alpha D^{\beta+e_i}$. For $M_1, \ldots, M_p \in \mathcal{D}_n$ define subspaces L of $\mathcal{D}_n^p$ and J of $\mathcal{D}_n$ as follows. For $P = \sum c_{\gamma\delta} x^\gamma D^\delta \in \mathcal{D}_n$, let $P^o = \sum c_{\gamma\delta} x^\gamma T^\delta$ and $P' = P - P^o$ be the induced operators on $\mathcal{D}_n$. Consider the map $f : \mathcal{D}_n^p \to \mathcal{D}_n$ given by $f(A) = \sum A^o_i x^{\alpha_i} D^{\beta_i}$. The kernel and the image of f admit direct complements L and J in $\mathcal{D}_n^p$ and $\mathcal{D}_n$ given by support conditions: Set $F = \bigcup_i (\alpha_i, \beta_i) + \mathbb{N}^{2n}$ and let $F = \bigcup_i F_i$ be a partition of F with $(\alpha_i, \beta_i) \in F_i$. Then

$$L = \{A \in \mathcal{D}_n^r, \text{ supp } A_i \subset F_i - (\alpha_i, \beta_i)\}$$
$$J = \{B \in \mathcal{D}_n, \text{ supp } B \subset \mathbb{N}^{2n} \setminus F\}.$$

We shall show that f is a sufficiently precise approximation of the $\mathcal{O}_n$-linear map $\mathcal{D}_n^p \to \mathcal{D}_n$ given by $A \to \sum A_i M_i$.

THEOREM. — *For (λ, μ) as above with $\mu \gg \lambda$ componentswise, the map*

$$u : L \oplus J \to \mathcal{D}_n, \ (A, B) \to \sum A_i M_i + B$$

is a bicontinuous compatible isomorphism. Assume that $\mu\beta_i > \mu\beta$ for all β in the support of M'_i. Then there are constants $s_0 > 0$ and $C > 0$ such that for all $E \in \mathcal{D}_n(s)$ the unique elements $A \in L(s)$ and $B \in J(s)$ with $E = \sum A_i M_i + B$ satisfy

$$\sum |A_i|_{st} \cdot |M_i|_{st} + |B|_{st} \leqslant C \cdot |E|_{st}$$

for $0 < s < s_0$ and $0 < t \leqslant s^2$.

Compatible means that u respects the filtrations of $\mathcal{D}_n$ by $\mathcal{D}_n(s)$ for sufficiently small s. Observe that the opposed map $(A, B) \to \sum M_i A_i + B$ is not compatible since derivations alter the radius of convergence of a series (however, it can be shown that the corresponding division is continuous).

The assumption on μ is satisfied in case that $(\alpha, \beta) < (\alpha', \beta')$ implies $(0, \ldots, 0, \beta) < (0, \ldots, 0, \beta')$ w.r.t. the monomial order on $\mathbb{N}^{2n}$. Without that assumption on μ, the estimate of the theorem holds for $0 < s < s_0$, $t = s$, $\lambda = \sigma$ and $\mu = \mathbf{k} - \rho$, $k \gg 0$, with C independent of k. Varying k, this case covers in particular the estimate of Theorem 3.1.2 of [MN].

Example. — We show that the assumption on μ cannot be omitted if t is allowed to approach 0 for fixed s. Take $M = \partial_x \partial_y - x \partial_x^2 - x \partial_y^2$ with

$(\lambda, \mu) = (\sigma_1, \sigma_2, k-\rho_1, k-\rho_2)$. Assume that $\rho_1 \neq \rho_2$ and that $|\rho_1 - \rho_2| < \sigma_1$. Then $A = \partial_x \partial_y$ is the initial monomial of M, and dividing A by M yields the remainder $R = x\partial_x^2 + x\partial_y^2$ with norm $|R|_{st} = s^{\lambda_1}(t^{-\mu_1} + t^{-\mu_2})$. There is no constant C such that this norm remains bounded by $C \cdot |A|_{st} = C \cdot t^{-\mu_1 - \mu_2}$ for fixed s and t going to 0.

Proof. — We show that u is bicontinuous by interpreting it as a perturbation of a purely combinatorial and bicontinuous bijective map v. As such u is shown to be bijective and its inverse will be given as v^{-1} times a geometric series in $v - u$. Bounding uniformly the norm of the restrictions of $v - u$ to $L(s) \oplus J(s)$ for $s > 0$ sufficiently small yields the required estimate for u^{-1}.

The map u is $\mathcal{O}_n$-linear and decomposes into $u = v + w_1 + w_2$ where

$$v(A, B) = \sum A_i^o x^{\alpha_i} D^{\beta_i} + B,$$

$$w_1(A, B) = \sum A_i' x^{\alpha_i} D^{\beta_i},$$

$$w_2(A, B) = \sum A_i M_i'.$$

The definition of L and J is chosen so that v is a bicontinuous compatible isomorphism. We shall show that there are positive constants s_0, r and C_1, C_2 such that for all $0 < s < s_0$, $t > 0$, $d \in \mathbb{N}$ and $E \in \mathcal{D}_n(s)_d$ we have

$$|(w_1 v^{-1})E|_{st} \leqslant C_1 \cdot |E|_{st} \cdot s^{-r} t^r$$

and

$$|(w_2 v^{-1})E|_{st} \leqslant C_2 \cdot |E|_{st} \cdot s^{-r} t^r.$$

Choosing s_0 sufficiently small and $0 < s < s_0$, $0 < t \leqslant s^2$ these inequalities imply that the geometric series $(\mathrm{Id} + (w_1 + w_2)v^{-1})^{-1}$ converges to a map on the space of differential operators of infinite order $\mathcal{D}_n(s)^\infty$. As $w_1 v^{-1}$ and $w_2 v^{-1}$ do not increase the order of a differential operator the series defines an automorphism of the Banach spaces $(\mathcal{D}_n(s)_d, | - |_{st})$ for each d and t, and in fact an automorphism of the locally convex space $\mathcal{D}_n(s)$. It follows that $u = (\mathrm{Id} + (w_1 + w_2)v^{-1})v^{-1}$ is an isomorphism from the spaces $v^{-1}\mathcal{D}_n(s)$ to $\mathcal{D}_n(s)$ whose inverse has norm $\leqslant C$ w.r.t. s and t for some constant $C > 0$. This is just the assertion of the theorem.

Choose $s_0 > 0$ such that M_i and all its derivatives $D^\delta M_i$, $\delta \in \mathbb{N}^n$, belong to $\mathcal{D}_n(s)$ for all i and all $0 < s < s_0$. Let $E \in \mathcal{D}_n$. Decompose E into $E = \sum_{i\gamma\delta} a_{\gamma\delta}^i x^{\alpha_i + \gamma} D^{\beta_i + \delta} + B$ with $B \in J$ and $A = (A_1, \ldots, A_p) \in L$,

$A_i = \sum_{\gamma\delta} a^i_{\gamma\delta} x^\gamma D^\delta$. Then
$$|E|_{st} \geqslant \left| \sum_{i\gamma\delta} a^i_{\gamma\delta} x^{\alpha_i+\gamma} D^{\beta_i+\delta} \right|_{st} = \sum_{i\gamma\delta} |a^i_{\gamma\delta}| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i+\gamma)} \cdot t^{-\mu(\beta_i+\delta)}.$$

The equality follows from the fact that all exponents are distinct. Set
$$B_{i\delta\gamma} = |a^i_{\gamma\delta}| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i+\gamma)} \cdot t^{-\mu(\beta_i+\delta)}.$$

(a) We estimate first the norm of $w_1 v^{-1}$. Denoting by $\alpha \leqslant \beta$ the componentwise order in $\mathbb{N}^n$, we have

$$|(w_1 v^{-1})E|_{st}$$
$$= \left| (w_1 v^{-1}) \left(\sum_{i\gamma\delta} a^i_{\gamma\delta} x^{\alpha_i+\gamma} D^{\beta_i+\delta} + B \right) \right|_{st}$$
$$= \left| w_1 \left(\left(\sum_{\gamma\delta} a^i_{\gamma\delta} x^\gamma D^\delta \right)_i , B \right) \right|_{st}$$
$$= \left| \sum_{i\gamma\delta} a^i_{\gamma\delta} x^\gamma (D^\delta - T^\delta) x^{\alpha_i} D^{\beta_i} \right|_{st}$$
$$= \left| \sum_{i\gamma\delta} a^i_{\gamma\delta} x^\gamma \sum_{0 < \varepsilon \leqslant \alpha_i, \delta} \binom{\delta}{\varepsilon} \cdot \frac{\alpha_i!}{(\alpha_i-\varepsilon)!} \cdot x^{\alpha_i-\varepsilon} \cdot D^{\beta_i+\delta-\varepsilon} \right|_{st}$$
$$\leqslant \sum_{i\gamma\delta} \sum_{0<\varepsilon\leqslant\alpha_i,\delta} |a^i_{\gamma\delta}| \cdot \binom{\alpha_i}{\varepsilon} \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i+\gamma-\varepsilon)} \cdot t^{-\mu(\beta_i+\delta-\varepsilon)}$$
$$\leqslant \sum_i \sum_{0<\varepsilon\leqslant\alpha_i} \sum_{\delta\geqslant\varepsilon} \sum_\gamma |a^i_{\gamma\delta}| \cdot 2^{|\alpha_i|} \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i+\gamma-\varepsilon)} \cdot t^{-\mu(\beta_i+\delta-\varepsilon)}.$$

Set
$$A_{i\gamma\delta\varepsilon} = |a^i_{\gamma\delta}| \cdot 2^{|\alpha_i|} \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i+\gamma-\varepsilon)} \cdot t^{-\mu(\beta_i+\delta-\varepsilon)}.$$

Then
$$\frac{|(w_1 v^{-1})E|_{st}}{|E|_{st}} \leqslant \frac{\sum_{i\varepsilon} \sum_{\gamma,\delta\geqslant\varepsilon} A_{i\gamma\delta\varepsilon}}{\sum_i \sum_{\gamma,\delta\geqslant 0} B_{i\delta\gamma}} \leqslant \sum_{i\varepsilon} \frac{\sum_{\gamma,\delta\geqslant\varepsilon} A_{i\gamma\delta\varepsilon}}{\sum_{\gamma\delta} B_{i\delta\gamma}}.$$

We have
$$\frac{\sum_{\gamma,\delta\geqslant\varepsilon} A_{i\gamma\delta\varepsilon}}{\sum_{\gamma\delta} B_{i\delta\gamma}}$$
$$= \frac{\sum_{\gamma\delta} |a^i_{\gamma\delta}| \cdot 2^{|\alpha_i|} \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda(\alpha_i+\gamma-\varepsilon)} \cdot t^{-\mu(\beta_i+\delta-\varepsilon)}}{\sum_{\gamma\delta} |a^i_{\gamma\delta}| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i+\gamma)} \cdot t^{-\mu(\beta_i+\delta)}}$$
$$= 2^{|\alpha_i|} \cdot s^{-\lambda\varepsilon} \cdot t^{\mu\varepsilon} \cdot \frac{\sum_{\gamma,\delta\geqslant\varepsilon} |a^i_{\gamma\delta}| \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta_i + \delta - \varepsilon|! \cdot s^{\lambda\gamma} \cdot t^{-\mu\delta}}{\sum_{\gamma\delta} |a^i_{\gamma\delta}| \cdot |\beta_i + \delta|! \cdot s^{\lambda\gamma} \cdot t^{-\mu\delta}}$$
$$\leqslant 2^{|\alpha_i|} \cdot s^{-\lambda\varepsilon} \cdot t^{\mu\varepsilon}.$$

The last inequality follows from

$$\frac{\delta! \cdot |\beta_i + \delta - \varepsilon|!}{(\delta - \varepsilon)! \cdot |\beta_i + \delta|!} \leqslant \frac{\delta! \cdot (\beta_i + \delta - \varepsilon)!}{(\delta - \varepsilon)! \cdot (\beta_i + \delta)!} = \binom{\delta}{\varepsilon} \cdot \binom{\beta_i + \delta}{\varepsilon}^{-1} \leqslant 1.$$

Here we use that for $\alpha, \beta \in \mathbb{N}^n$ with $\alpha \leqslant \beta$ one has $\frac{|\alpha|!}{\alpha!} \leqslant \frac{|\beta|!}{\beta!}$. Combining the above gives

$$\frac{|(w_1 v^{-1}) E|_{st}}{|E|_{st}} \leqslant \sum_i \sum_{0 < \varepsilon \leqslant \alpha_i} 2^{|\alpha_i|} \cdot s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \leqslant C_1' \cdot \sum_i \sum_{0 < \varepsilon \leqslant \alpha_i} s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon}$$

for some constant $C_1' > 0$. Choose r such that $\lambda \varepsilon \leqslant r \leqslant \mu \varepsilon$ for all $0 < \varepsilon \leqslant \alpha_i$ and all i. This is possible since $\mu \gg \lambda$. We obtain for suitable $C_1 > 0$ that

$$\frac{|(w_1 v^{-1}) E|_{st}}{|E|_{st}} \leqslant C_1' \cdot \sum_i \sum_{0 < \varepsilon \leqslant \alpha_i} s^{-\lambda \varepsilon} \cdot t^{\mu \varepsilon} \leqslant C_1 \cdot s^{-r} \cdot t^r.$$

(b) We estimate the norm of $w_2 v^{-1}$.

$$|(w_2 v^{-1}) E|_{st}$$

$$= \left| (w_2 v^{-1}) \left(\sum_{\imath \gamma \delta} a^i_{\gamma \delta} x^{\alpha_i + \gamma} D^{\beta_i + \delta} + B \right) \right|_{st}$$

$$= \left| w_2 \left(\left(\sum_{\gamma \delta} a^i_{\gamma \delta} x^\gamma D^\delta \right)_\imath, B \right) \right|_{st}$$

$$= \left| \sum_{\imath \gamma \delta} (a^\imath_{\gamma \delta} x^\gamma D^\delta) M'_\imath \right|_{st}$$

$$= \left| \sum_{\imath \gamma \delta} (a^\imath_{\gamma \delta} x^\gamma D^\delta) \left(\sum_{\alpha \beta} c^\imath_{\alpha \beta} x^\alpha D^\beta \right) \right|_{st}$$

$$= \left| \sum_{\imath \gamma \delta} \sum_{\alpha \beta} a^\imath_{\gamma \delta} c^\imath_{\alpha \beta} x^\gamma D^\delta (x^\alpha D^\beta) \right|_{st}$$

$$= \left| \sum_{\imath \gamma \delta} \sum_{\alpha \beta} a^i_{\gamma \delta} c^\imath_{\alpha \beta} x^\gamma \sum_{0 \leqslant \varepsilon \leqslant \alpha, \delta} \binom{\delta}{\varepsilon} \cdot \frac{\alpha!}{(\alpha - \varepsilon)!} \cdot x^{\alpha - \varepsilon} \cdot D^{\beta + \delta - \varepsilon} \right|_{st}$$

$$= \left| \sum_{\imath \gamma \delta} \sum_{\alpha \beta} \sum_\varepsilon a^\imath_{\gamma \delta} \cdot c^\imath_{\alpha \beta} \cdot \frac{\delta!}{(\delta - \varepsilon)!} \cdot \binom{\alpha}{\varepsilon} \cdot x^{\alpha + \gamma - \varepsilon} \cdot D^{\beta + \delta - \varepsilon} \right|_{st}$$

$$\leqslant \sum_{\imath \gamma \delta} \sum_{\alpha \beta} \sum_\varepsilon |a^\imath_{\gamma \delta}| \cdot |c^\imath_{\alpha \beta}| \cdot \frac{\delta!}{(\delta - \varepsilon)!} \cdot \binom{\alpha}{\varepsilon} \cdot |\beta + \delta - \varepsilon|! \cdot s^{\lambda(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}$$

$$= \sum_{\imath \alpha \beta} \sum_{0 \leqslant \varepsilon \leqslant \alpha} |c^\imath_{\alpha \beta}| \cdot \binom{\alpha}{\varepsilon} \cdot \sum_{\delta \geqslant \varepsilon} \sum_\gamma |a^\imath_{\gamma \delta}| \cdot \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta + \delta - \varepsilon|! \cdot s^{\lambda(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}.$$

The inequality need not be an equality since certain $\alpha + \gamma - \varepsilon$, respectively $\beta + \delta - \varepsilon$, may coincide. Set

$$C_{i\alpha\beta\gamma\delta\varepsilon} = |a^i_{\gamma \delta}| \cdot \frac{\delta!}{(\delta - \varepsilon)!} \cdot |\beta + \delta - \varepsilon|! \cdot s^{\lambda(\alpha + \gamma - \varepsilon)} \cdot t^{-\mu(\beta + \delta - \varepsilon)}.$$

Then

$$\frac{|(w_2 v^{-1})E|_{st}}{|E|_{st}}$$

$$\leqslant \frac{\sum_{i\alpha\beta} \sum_{\varepsilon} |c^i_{\alpha\beta}| \cdot \binom{\alpha}{\varepsilon} \cdot \sum_{\delta \geqslant \varepsilon} \sum_{\gamma} C_{i\alpha\beta\gamma\delta\varepsilon}}{\sum_i \sum_{\delta \geqslant 0} \sum_{\gamma} B_{i\delta\gamma}}$$

$$\leqslant \sum_{i\alpha\beta\varepsilon} |c^i_{\alpha\beta}| \cdot \binom{\alpha}{\varepsilon} \cdot \frac{\sum_{\gamma,\delta \geqslant \varepsilon} C_{i\alpha\beta\gamma\delta\varepsilon}}{\sum_{\gamma\delta} B_{i\delta\gamma}}$$

$$= \sum_{i\alpha\beta\varepsilon} |c^i_{\alpha\beta}| \cdot \binom{\alpha}{\varepsilon} \cdot \frac{\sum_{\gamma,\delta \geqslant \varepsilon} |a^i_{\gamma\delta}| \cdot \frac{\delta!}{(\delta-\varepsilon)!} \cdot |\beta + \delta - \varepsilon|! \cdot s^{\lambda(\alpha+\gamma-\varepsilon)} \cdot t^{-\mu(\beta+\delta-\varepsilon)}}{\sum_{\gamma\delta} |a^i_{\gamma\delta}| \cdot |\beta_i + \delta|! \cdot s^{\lambda(\alpha_i+\gamma)} \cdot t^{-\mu(\beta_i+\delta)}}$$

$$\leqslant C'_2 \cdot \sum_{i\alpha\beta} \sum_{0 \leqslant \varepsilon \leqslant \alpha} |c^i_{\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha-\alpha_i-\varepsilon)} \cdot t^{-\mu(\beta-\beta_i-\varepsilon)} =: R$$

for some constant $C'_2 > 0$ and using the assumption $|\beta| \leqslant |\beta_i|$. For each β which is an exponent of D in the monomials of M'_i let $(\alpha_{i\beta}, \beta)$ be an element of the support of M'_i with minimal first component: $\lambda\alpha_{i\beta} \leqslant \lambda\alpha$ for all α with $(\alpha, \beta) \in \operatorname{supp} M'_i$. Then

$$R \leqslant C'_2 \cdot \sum_{i\alpha\beta} \sum_{0 \leqslant \varepsilon \leqslant \alpha} |c^i_{\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{-\lambda\varepsilon} \cdot t^{\mu\varepsilon} \cdot s^{\lambda(\alpha-\alpha_{i\beta})} \cdot s^{\lambda(\alpha_{i\beta}-\alpha_i)} \cdot t^{-\mu(\beta-\beta_i)}$$

$$\leqslant C'_2 \cdot s^{-r} \cdot t^r \cdot \sum_{\varepsilon} s^{-\lambda\varepsilon} t^{\mu\varepsilon} \cdot \sum_{i\beta} \sum_{\alpha} |c^i_{\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha-\alpha_{i\beta})}$$

where $r > 0$ is chosen such that $\lambda(\alpha_i - \alpha_{i\beta}) \leqslant r \leqslant \mu(\beta_i - \beta)$ for every i and β (here the assumption $\mu(\beta_i - \beta) > 0$ is used). The last series converges and remains bounded for s sufficiently small because of $\lambda(\alpha - \alpha_{i\beta}) \geqslant 0$. This shows

$$\frac{|(w_2 v^{-1})E|_{st}}{|E|_{st}} \leqslant C_2 \cdot s^{-r} \cdot t^r$$

for $C_2 > 0$ suitable and concludes the proof of the theorem.

In case $\mu(\beta_i - \beta)$ is arbitrary but t equals s, the first estimate for R holds again, and the second inequality is

$$\leqslant C'_2 \cdot s^a \cdot \sum_{\varepsilon} s^{\mu\varepsilon-\lambda\varepsilon} \cdot \sum_{i\beta} \sum_{\alpha} |c^i_{\alpha\beta}| \cdot 2^{|\alpha|} \cdot s^{\lambda(\alpha-\alpha_{i\beta})}$$

where a is a positive constant such that

$$0 < a \leqslant \lambda(\alpha_{i\beta} - \alpha_i) - \mu(\beta - \beta_i)$$

for all i and β. Then

$$\frac{|(w_2 v^{-1})E|_{ss}}{|E|_{ss}} \leqslant C_2 \cdot s^a$$

for $0 < s < s_0$ and $C_2 > 0$ suitable but independent of $k \gg 0$ (since the series $\sum_\varepsilon s^{\mu\varepsilon - \lambda\varepsilon} = \sum_\varepsilon s^{k|\varepsilon|} s^{-(\rho + \sigma)\varepsilon}$ remains bounded for $k \gg 0$). $\qquad\square$

As in [MN], 4.1, we can follow Serre's idea in the case of the completion of a local noetherian commutative ring and deduce from the theorem above a very short proof of the faithfully flatness of the extension $\mathcal{D}_n \subset \mathcal{D}_n^\infty$, originally proven by Sato-Kashiwara-Kawai by using microlocal techniques.

Addendum.

Fix a monomial order ε in $\mathbb{N}^{2n}$. We prove that for finitely many differential operators M_i there exists $(\lambda, \mu) \in \mathbb{Q}_+^{2n}$, with $\lambda \ll \mu$, such that for each i there is a unique (α_i, β_i) in the support of M_i with $\lambda\alpha_i - \mu\beta_i$ minimal and such that $x^{\alpha_i} D^{\beta_i}$ is the initial monomial of M_i w.r.t. ε.

Recall that on any finite subset of $\mathbb{N}^n$ every monomial order can be defined by an $\mathbb{N}$-linear map $\eta : \mathbb{N}^n \to \mathbb{N}$ by pulling back the natural order on $\mathbb{N}$. Identify η with a vector in $\mathbb{N}^n$. Let $(\sigma, \rho) \in \mathbb{N}_+^{2n}$ be weights which induce the same order as ε on a sufficiently big subset of $\mathbb{N}^{2n}$ [including the finitely many monomials of M_i which have minimal x-exponent and maximal D-exponent w.r.t. the componentwise order on $\mathbb{N}^n$]. Set $\lambda = \sigma$ and $\mu = \mathbf{k} - \rho$, where k is a positive integer and $\mathbf{k} = (k, \ldots, k) \in \mathbb{N}^n$. Then $\lambda\alpha - \mu\beta = -k \cdot |\beta| + (\sigma\alpha + \rho\beta)$. For k sufficiently large and for each i the pair (α, β) in the support of M_i with $\lambda \cdot \alpha - \mu \cdot \beta$ minimal satisfies

$|\beta|$ is maximal, i.e. (α, β) belongs to the support of the symbol of M_i.

(α, β) is minimal in the support of the symbol of M_i w.r.t. (σ, ρ).

This implies that the monomial of M_i with exponent (α, β) minimal w.r.t. $\lambda \cdot \alpha - \mu \cdot \beta$ coincides with the minimal monomial of the symbol of M_i w.r.t. ε. By taking k sufficiently large we obtain $\lambda \ll \mu$. For ε the inverse graded order as in [C] let $(\pi, \tau) \in \mathbb{N}_+^{2n}$ be weights which define in the chosen finite subset of $\mathbb{N}^{2n}$ the inverse lexicographic order (not the graded one). Then

$$\lambda = \mathbf{k} + \pi \quad \text{and} \quad \mu = \mathbf{k}^2 - \tau$$

will work for k sufficiently large.

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