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## ON NORMAL ABELIAN SUBGROUPS IN PARABOLIC GROUPS

by Gerhard RÖHRLE (\*)

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### 1. Introduction.

Throughout,  $G$  denotes a (connected) reductive algebraic group defined over an algebraically closed field  $k$  of characteristic  $p \geq 0$  and  $P$  is a parabolic subgroup of  $G$  with unipotent radical  $P_u$ . The aim of this note is the following result.

**THEOREM 1.1.** — *Let  $G$  be a reductive algebraic group,  $P$  a parabolic subgroup of  $G$ , and  $A$  a closed connected normal subgroup of  $P$  in  $P_u$ . If  $A$  is abelian, then  $P$  has finitely many orbits on  $A$ .*

The particular case when  $A$  is in the center of  $P_u$  is well-known. Then the action factors through a Levi subgroup of  $P$ . In characteristic 0 the finiteness follows from a result of Vinberg [41, § 2] on gradings of Lie algebras (see also Kac [15]) and in general from work of Richardson [28, § 3]. For a detailed account of the orbit structure in this situation, see [24] and [29, § 2, § 5].

Observe that for abelian  $P$ -invariant sub-factors in  $P_u$ , the analogous statement of the theorem is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on the unipotent radical, or its Lie algebra, *e.g.*, see [25], [26], [30], and [31]. Examples in this context also show that a parabolic

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subgroup may have an infinite number of orbits on a normal subgroup of nilpotency class two, cf. [9], [16].

The *modality* of the action of  $P$  on the normal subgroup  $A$  is the maximal number of parameters upon which a family of  $P$ -orbits on  $A$  depends; likewise for the adjoint action of  $P$  on the Lie algebra of  $A$ , cf. [26]. The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [26]. Apart from [26] there are several recent articles related to this subject. For instance, all parabolic subgroups  $P$  of classical algebraic groups with a finite number of orbits on  $P_u$  are determined in [12] and [13]. Similar results for exceptional groups are obtained in [14]. In [8] all such  $P$  in  $\mathrm{GL}_n(k)$  are classified with a finite number of orbits on a given term of the lower central series of  $P_u$ . More generally, in [25] and [32] the modality of the action of  $P$  on the Lie algebra of  $P_u$  is investigated for any reductive  $G$ .

The proof of Theorem 1.1 readily reduces to the case when  $G$  is simple,  $P$  is a Borel subgroup  $B$  of  $G$ , and  $A$  is a maximal closed connected normal abelian subgroup of  $B$ . In Section 3 we classify all such  $A$ , up to  $G$ -conjugacy (Theorem 3.1), and in Section 5 we show in each instance that  $B$  acts on  $A$  with a finite number of orbits.

Our proof of Theorem 1.1 uses an extension of the classification of spherical Levi subgroups of reductive groups to arbitrary characteristic due to Brundan [4, § 4] (see Section 4). From that we immediately obtain a proof of those cases of Theorem 1.1 where  $A$  is contained in the unipotent radical of a parabolic subgroup whose Levi factor is spherical in  $G$ . In the cases where we cannot appeal to spherical Levi subgroups directly, a construction from [1] allows us to apply these results partially.

In the two final sections we discuss the situation for the adjoint action of  $P$  on abelian ideals in the Lie algebra of  $P_u$  as well as a connection between Theorem 3.1 and Mal'cev's classification of abelian subalgebras of the Lie algebra of  $G$  of maximal dimension [21].

Both, our classification of the maximal closed connected normal abelian subgroups  $A$  of  $B$ , as well as the fact that in each of these instances  $B$  operates on  $A$  with a finite number of orbits, are obtained in case studies. It would be highly desirable to have a uniform proof of Theorem 1.1 free of case analysis, even for the expense of some characteristic restrictions.

## 2. Notation and preliminaries.

We denote the Lie algebra of  $G$  by  $\text{Lie } G$  or  $\mathfrak{g}$  and the identity element of  $G$  by  $e$ ; likewise for subgroups of  $G$ . Let  $T$  be a fixed maximal torus in  $G$  and  $\Psi = \Psi(G)$  the set of roots of  $G$  with respect to  $T$ . Fix a Borel subgroup  $B$  of  $G$  containing  $T$  and let  $\Sigma = \{\sigma_1, \sigma_2, \dots\}$  be the set of simple roots of  $\Psi$  defined by  $B$  such that the positive integral span of  $\Sigma$  in  $\Psi$  is  $\Psi^+ = \Psi(B)$ . The highest (long) root in  $\Psi$  is denoted by  $\varrho$ . If all roots in  $\Psi$  are of the same length, they are all called long. A subset  $I$  of  $\Psi^+$  is an *ideal* in  $\Psi^+$  (see [38, p. 24]) provided  $I$  is closed under addition by elements from  $\Psi^+$ . For a root  $\beta$  of  $G$  we denote by  $U_\beta$  the corresponding one-parameter unipotent subgroup of  $G$  normalized by  $T$ , and the root subspace  $\text{Lie } U_\beta$  of  $\mathfrak{g}$  by  $\mathfrak{u}_\beta$ . The members  $U_\beta(\xi)$ , where  $\xi \in k$ , of  $U_\beta$  are called *root elements*.

Suppose that  $G$  is simple (over its center). A prime is said to be *bad* for  $G$  if it divides the coefficient of a simple root in  $\varrho$ , else it is called *good* for  $G$  [37, § I,4]. Furthermore, we say that a prime is *very bad* for  $G$  if it divides a structure constant of the Chevalley commutator relations for  $G$ . Thus, if  $p$  is very bad for  $G$ , there are degeneracies in these relations. This only occurs if  $p = 2$  and  $G$  is of type  $B_r$ ,  $C_r$ ,  $F_4$ , or  $G_2$ , or  $p = 3$  and  $G$  is of type  $G_2$ . The same notions apply to reductive groups by means of simple components [39, 3.6].

We may assume that each parabolic subgroup  $P$  of  $G$  considered contains  $B$ .

Let  $N$  be a closed connected normal subgroup of  $P$  in  $P_u$ . Since  $N$  is normalized by  $T \subset P$ , i.e.,  $N$  is  $T$ -regular [11], the root spaces of  $\mathfrak{n}$  relative to  $T$  are also root spaces of  $\mathfrak{g}$  relative to  $T$ , and the set of roots of  $N$  with respect to  $T$ , denoted by  $\Psi(N)$ , is a subset of  $\Psi$ . Suppose that  $\Psi(N)$  is closed under addition in  $\Psi$ . Note that this is automatically satisfied provided  $p$  is not very bad for  $G$ . Then  $\mathfrak{n} = \bigoplus \mathfrak{u}_\beta$  ( $\beta \in \Psi(N)$ ) and consequently,  $N = \prod U_\beta$ , where the product is taken in some fixed order over  $\Psi(N)$ . The *support* of an element  $x$  in  $N$ , denoted by  $\text{supp } x$ , consists of all roots  $\beta$  for which the projection  $N \rightarrow U_\beta$  is nontrivial when evaluated at  $x$ .

By the *shape of a root*  $\beta = \sum n_\sigma(\beta)\sigma$  ( $\sigma \in \Sigma$ ) *relative to*  $P$ , we mean the sub-sum over the elements of  $\Sigma(P_u) = \Psi(P_u) \cap \Sigma$ , and by the *level of*  $\beta$  *relative to*  $P$ , the sub-sum of the coefficients  $n_\sigma(\beta)$  over the same set  $\Sigma(P_u)$ , cf. [1].

The descending central series of  $P_u$  is defined as usual by  $\mathcal{C}^0 P_u := P_u$  and  $\mathcal{C}^{i+1} P_u := (\mathcal{C}^i P_u, P_u)$  for  $i \geq 0$ . Since  $P_u$  is nilpotent, the smallest integer  $m$  such that  $\mathcal{C}^m P_u = \{e\}$  is the class of nilpotency of  $P_u$ , *i.e.*, the *length* of this series, and is also denoted by  $\ell(P_u)$ . If  $p$  is not a very bad prime for  $G$ , then  $\Psi(\mathcal{C}^i P_u)$  consists precisely of all roots whose  $P$ -level is at least  $i + 1$ , see [1].

Throughout, we use the labeling of the Dynkin diagram of  $G$  (*i.e.* of  $\Sigma$ ) as well as the notation for roots in systems of exceptional type as in Bourbaki [3]. Our general reference for algebraic groups is Borel's book [2].

### 3. The maximal normal abelian subgroups of Borel subgroups.

In this section we determine all maximal closed connected normal abelian subgroups  $A$  of our fixed Borel subgroup  $B$  of  $G$  and record them in the subsequent table. Here we specify the roots  $\alpha$  such that  $A$  is the normal closure in  $B$  of the corresponding root subgroups  $U_\alpha$ . The fact that  $A$  is abelian follows either from the observation that the sum of two roots in  $\Psi(A)$  is not a root, because it exceeds  $\varrho$  in some coefficient, and thus, by the commutator relations,  $A$  is commutative, or else because  $p$  is a very bad prime for  $G$  leading to commutation degeneracies. As indicated in the table, some extra cases do occur for very bad primes.

The simple roots  $\sigma_i$  are labeled as in [3]. Moreover, we use the following abbreviations: in type  $B_r$  set

$$\beta_i = \sigma_1 + \cdots + \sigma_i, \quad \gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_r, \quad \delta_i = \sigma_i + \cdots + \sigma_r,$$

where  $2 \leq i \leq r$ , and finally  $\eta = \beta_r + \sigma_r$ . Similarly, for type  $D_r$  we define

$$\beta_i = \sigma_1 + \cdots + \sigma_i, \quad \gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_{r-2} + \sigma_{r-1} + \sigma_r$$

for  $3 \leq i \leq r - 2$  and

$$\beta = \beta_{r-2} + \sigma_{r-1}, \quad \gamma = \beta_{r-2} + \sigma_r, \quad \delta = \sigma_{r-2} + \sigma_{r-1} + \sigma_r.$$

The normalizer of  $A$  in  $G$  is a parabolic subgroup of  $G$ , since it contains  $B$ . In the third column of Table 1 we indicate the set of simple roots of the standard Levi subgroup of  $N_G(A)$ . Here the notation  $\{\sigma'_1, \sigma'_2, \dots\}$  simply means  $\Sigma \setminus \{\sigma_1, \sigma_2, \dots\}$ . Finally, we list  $\dim A$  in each instance.

**THEOREM 3.1.** — *Let  $G$  be a simple algebraic group. Every maximal closed connected normal abelian subgroup of the Borel subgroup  $B$  of  $G$  is listed in Table 1.*

*Proof.* — First we assume that  $p$  is not very bad for  $G$ . Then, our aim to determine each maximal closed connected normal abelian subgroup  $A$  of  $B$  is equivalent to the purely combinatorial task to determine all maximal abelian ideals  $\Psi(A)$  in  $\Psi(B)$  (i.e., all those ideals  $I$  of  $\Psi(B)$  which are maximal with respect to the property that no two roots in  $I$  sum up to a root in  $\Psi$ ). If  $A$  is the normal closure in  $B$  of a single root subgroup  $U_\alpha$ , then  $2\alpha$  exceeds  $\varrho$  in some coefficient, because of the commutator relations for root subgroups in  $A$ . This quickly leads to a complete list of all the maximal  $\Psi(A)$ 's of this nature. If  $A$  is the normal closure of two distinct root subgroups, then the normal closure of each one of them is abelian and no sum of any two roots from the supports of these two subgroups is again a root. One checks that while in type  $A_r$  or  $C_r$  any such  $A$  is already contained in one of the first kind, in type  $B_r$  and  $D_r$  there are new occurrences of the second type which are not contained in ones of the first kind. The maximal ones then are easily determined which lead to the families in the second entry for  $B_r$  and the third one for  $D_r$ . As indicated, there are also new cases here for the exceptional types. Abelian normal subgroups which are the normal closure in  $B$  of three distinct root subgroups and which are not already contained in one of the first two kinds only arise in type  $D_r$ ,  $E_6$ ,  $E_7$ , and  $E_8$ . There are no maximal incidences for any type when  $\Psi(A)$  is generated by four or more distinct roots.

For the exceptional groups these records were obtained with the aid of a computer algorithm which, for any given  $G$ , computes all maximal abelian ideals of  $\Psi(B)$ . Consequently, as the rank of  $G$  is bounded in these cases, this yields the desired subgroups  $A$  in  $B$  in these cases.

Now we consider the situation when  $p$  is a very bad prime for  $G$ . Here we can only have additional cases when there are two root lengths in  $\Psi$ . Since in characteristic 2 the simple groups of type  $B_r$  and  $C_r$  are isomorphic as abstract groups [38, Thm 28], it suffices to only list the new occurrences for type  $B_r$ . In type  $C_r$  we only record the single generic case. For  $G_2$  there is only one additional occurrence for  $p = 3$  (none for  $p = 2$ ). One checks these remaining events directly.

Note, if  $I$  is the ideal in  $\Psi(B)$  generated by some positive roots, then  $U_\gamma$ , for some  $\gamma \in I$ , need not be contained in the normal closure of the corresponding root subgroups in  $B$ , because of commutator degeneracies in

the presence of very bad primes. This necessitates that in some of the cases in Table 1 we have to include additional generating roots. For instance, this is the case for  $G_2$  and in the third instance for  $F_4$ . This is not required in the third case for  $B_r$ , as the structure constants of the commutator of the root subgroups relative to the simple roots in a group of type  $B_2$  are all equal to  $\pm 1$ , *e.g.*, see [38, Lemma 33].

The last entry for  $B_r$  is the only event when  $\Psi(A)$  fails to be an ideal in  $\Psi^+$ . Observe that the roots  $\sigma_{i-1}$  and  $\delta_i$  form a basis of a subsystem of  $\Psi$  of type  $B_2$ . Thus by [38, Lemma 33]  $\gamma_i = \sigma_{i-1} + 2\delta_i$  is in  $\Psi(A)$  in this instance. Whence, if  $p = 2$ , the subgroup from the second entry for  $B_r$  is properly contained in the one from the fourth case (for a fixed  $i$ ).  $\square$

*Remark 3.2.* — Suppose that  $p$  is not a very bad prime for  $G$ . Let  $A$  be as in Table 1 below. Let  $N_G(A) = P = LP_u$  with standard Levi subgroup  $L$ . The simple roots  $\Sigma(L)$  canonically define a grading of  $\mathfrak{g}$  as follows [15]: Define a function  $d: \Psi \rightarrow \mathbb{Z}$  by setting  $d(\sigma) = 0$  if  $\sigma$  is in  $\Sigma(L)$  and  $d(\sigma) = 1$  if  $\sigma$  is in  $\Sigma \setminus \Sigma(L)$ , and extend  $d$  linearly to all of  $\Psi$ . Then we define

$$\mathfrak{g}(i) := \begin{cases} \bigoplus_{d(\alpha)=i} \mathfrak{u}_\alpha, & \text{for } i \neq 0; \\ \mathfrak{t} \oplus \bigoplus_{d(\alpha)=0} \mathfrak{u}_\alpha, & \text{for } i = 0. \end{cases}$$

Thus we have  $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$  and moreover,

$$\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i) \quad \text{and} \quad \mathfrak{p}_u = \bigoplus_{i > 0} \mathfrak{g}(i).$$

D. Panyushev observed that for each  $A$  from Table 1 below  $d(\varrho)$  is odd and for  $m = \lceil \frac{1}{2}d(\varrho) \rceil + 1$  we have  $\text{Lie } A = \bigoplus_{i \geq m} \mathfrak{g}(i)$ . Using the description of  $P$  furnished in the third column in Table 1, the value of  $d(\varrho)$  is readily determined.

*Remark 3.3.* — It is interesting to observe that if  $p$  is not very bad for  $G$ , then the number of closed connected maximal abelian normal subgroups of  $B$  equals the number of long simple roots of  $G$ .

Unfortunately, our proof of Theorem 3.1, involving case by case considerations, is less than satisfactory. It would be very desirable to have a uniform proof of this result given that  $p$  is not a very bad prime for  $G$ .

| Type of $G$ | $A$                                                                                 | $N_G(A)$                              | $\dim A$                   |
|-------------|-------------------------------------------------------------------------------------|---------------------------------------|----------------------------|
| $A_r$       | $\sigma_i \ (1 \leq i \leq r)$                                                      | $\sigma'_i$                           | $i(r-i+1)$                 |
| $B_r$       | $\sigma_1$                                                                          | $\sigma'_1$                           | $2r-1$                     |
|             | $\beta_i, \gamma_i \ (3 \leq i \leq r) \ (p \neq 2)$                                | $\sigma'_1, \sigma'_i$                | $\frac{1}{2}(4r+i^2-5i+2)$ |
|             | $\sigma_r \ (p=2)$                                                                  | $\sigma'_r$                           | $\frac{1}{2}(r^2+r)$       |
|             | $\beta_i, \delta_i, \eta \ (1 < i < r) \ (p=2)$                                     | $\sigma'_1, \sigma'_i$                | $\frac{1}{2}(4r+i^2-3i)$   |
| $C_r$       | $\sigma_r$                                                                          | $\sigma'_r$                           | $\frac{1}{2}(r^2+r)$       |
| $D_r$       | $\sigma_1$                                                                          | $\sigma'_1$                           | $2r-2$                     |
|             | $\sigma_{r-1}$ , or $\sigma_r$                                                      | $\sigma'_{r-1}$ , or $\sigma'_r$      | $\frac{1}{2}(r^2-r)$       |
|             | $\beta_i, \gamma_i \ (3 \leq i \leq r-2)$                                           | $\sigma'_1, \sigma'_i$                | $\frac{1}{2}(4r-5i+i^2)$   |
|             | $\beta, \gamma, \delta$                                                             | $\sigma'_1, \sigma'_{r-1}, \sigma'_r$ | $\frac{1}{2}(r^2-3r+6)$    |
| $E_6$       | $\sigma_1$ , or $\sigma_6$                                                          | $\sigma'_1$ , or $\sigma'_6$          | 16                         |
|             | $\begin{smallmatrix} 01210 \\ 1 \end{smallmatrix}$                                  | $\sigma'_4$                           | 11                         |
|             | $\begin{smallmatrix} 11110, 01221 \\ 0 \quad 1 \end{smallmatrix}$                   | $\sigma'_1, \sigma'_5$                | 13                         |
|             | $\begin{smallmatrix} 01111, 12210 \\ 0 \quad 1 \end{smallmatrix}$                   | $\sigma'_3, \sigma'_6$                | 13                         |
|             | $\begin{smallmatrix} 11111, 01211, 11210 \\ 0 \quad 1 \quad 1 \end{smallmatrix}$    | $\sigma_2, \sigma_3, \sigma_5$        | 12                         |
| $E_7$       | $\sigma_7$                                                                          | $\sigma'_7$                           | 27                         |
|             | $\begin{smallmatrix} 122100 \\ 1 \end{smallmatrix}$                                 | $\sigma'_3$                           | 17                         |
|             | $\begin{smallmatrix} 012210 \\ 1 \end{smallmatrix}$                                 | $\sigma'_5$                           | 20                         |
|             | $\begin{smallmatrix} 012221, 122110 \\ 1 \quad 1 \end{smallmatrix}$                 | $\sigma'_3, \sigma'_6$                | 18                         |
|             | $\begin{smallmatrix} 012111, 123210 \\ 1 \quad 1 \end{smallmatrix}$                 | $\sigma'_4, \sigma'_7$                | 20                         |
|             | $\begin{smallmatrix} 001111, 123210 \\ 1 \quad 2 \end{smallmatrix}$                 | $\sigma'_2, \sigma'_7$                | 22                         |
|             | $\begin{smallmatrix} 012211, 122210, 122111 \\ 1 \quad 1 \quad 1 \end{smallmatrix}$ | $\sigma'_3, \sigma'_5, \sigma'_7$     | 19                         |

Table 1 (first part): the maximal normal abelian subgroups of Borel subgroups

| Type of $G$ | $A$                                  | $N_G(A)$                          | $\dim A$ |
|-------------|--------------------------------------|-----------------------------------|----------|
| $E_8$       | $0122221$<br>1                       | $\sigma'_7$                       | 29       |
|             | $1233210$<br>1                       | $\sigma'_5$                       | 34       |
|             | $1232100$<br>2                       | $\sigma'_2$                       | 36       |
|             | $1122221, 2343210$<br>1 2            | $\sigma'_1, \sigma'_7$            | 30       |
|             | $1222221, 1343210$<br>1 2            | $\sigma'_3, \sigma'_7$            | 31       |
|             | $1232221, 1243210$<br>1 2            | $\sigma'_4, \sigma'_7$            | 32       |
|             | $1233321, 1232210$<br>1 2            | $\sigma'_2, \sigma'_6$            | 34       |
|             | $1233221, 1232221, 1233210$<br>1 2 2 | $\sigma'_2, \sigma'_5, \sigma'_7$ | 33       |
| $F_4$       | 1220                                 | $\sigma'_2$                       | 8        |
|             | 1221, 0122 ( $p \neq 2$ )            | $\sigma_1, \sigma_3$              | 9        |
|             | 0121, 0122 ( $p = 2$ )               | $\sigma_1, \sigma_2$              | 11       |
|             | 1111, 0122 ( $p = 2$ )               | $\sigma_2, \sigma_3$              | 11       |
| $G_2$       | 21 ( $p \neq 3$ )                    | $\sigma_2$                        | 3        |
|             | 11, 21 ( $p = 3$ )                   | $\emptyset$                       | 4        |

Table 1 (second part): the maximal normal abelian subgroups of Borel subgroups

#### 4. Spherical Levi subgroups.

The following basic result is due to Brion [4] and Vinberg [42] in characteristic 0 and in arbitrary characteristic to Knop [17, 2.6].

**THEOREM 4.1.** — *Let  $G$  be a reductive algebraic group and  $B$  a Borel subgroup of  $G$ . Let  $X$  be an irreducible  $G$ -variety admitting a dense  $B$ -orbit. Then  $B$  has finitely many orbits on  $X$ .*

A closed subgroup  $H$  of  $G$  is called *spherical* if  $H$  has a dense orbit on  $G/B$ , or equivalently, if there is a dense  $B$ -orbit on  $G/H$ , or equivalently, by Theorem 4.1, if there is a finite number of  $B$ -orbits on  $G/H$ . Concerning

recent results on spherical subgroups in positive characteristic, consult [7]. The associated *spherical varieties*  $G/H$  have been studied extensively in the literature; see [6] for a survey.

In characteristic 0 all reductive spherical subgroups have been classified in [20], [5], and [23]. When  $p \neq 2$  many instances are known as centralizers of involutions by a result of Springer [36]. Brundan's method of "integral embeddings" [7, § 2] and the representation theoretic fact that Levi subgroups of  $G$  are good filtration subgroups (see [10], [22]) enable him to extend the classification of spherical Levi subgroups from characteristic zero [20] to arbitrary characteristic [7, Thm 4.1]:

**THEOREM 4.2.** — *Let  $L$  be a Levi subgroup of  $G$ . Let  $G = \prod_{i=1}^r G_i$  as a commuting product of simple factors and  $L_i := L \cap G_i$ . Then  $L$  is spherical if and only if, for each  $i$ , either  $L_i = G_i$ , or  $(G_i, L'_i)$  is one of  $(A_r, A_m A_{r-m-1})$ ,  $(B_r, B_{r-1})$ ,  $(B_r, A_{r-1})$ ,  $(C_r, C_{r-1})$ ,  $(C_r, A_{r-1})$ ,  $(D_r, D_{r-1})$ ,  $(D_r, A_{r-1})$ ,  $(E_6, D_5)$ , or  $(E_7, E_6)$ .*

Theorem 4.2 and the next result [5, Prop. I.1] (or [7, Lemma 4.2]) yield a classification of all parabolic subgroups  $P = LP_u$  of  $G$  for which a Borel subgroup  $B_L$  of  $L$  has a dense orbit on  $P_u$ , whence finitely many orbits thanks to Theorem 4.1.

**LEMMA 4.3.** — *Let  $P = LP_u$  be a parabolic subgroup of  $G$  and  $B_L$  a Borel subgroup of  $L$ . Then  $L$  is spherical in  $G$  if and only if there is a dense  $B_L$ -orbit on  $P_u$ .*

**Remark 4.4.** — A Levi subgroup  $L$  of  $P$  is spherical in  $G$  if and only if either  $P_u$  is abelian, or  $p \neq 2$  and one of the pairs  $(G_i, L'_i)$  as in Theorem 4.2 equals  $(B_r, A_{r-1})$  or  $(C_r, C_{r-1})$ . This is immediate from Theorem 4.2 and the well-known instances when  $P_u$  is abelian, *e.g.*, see [29, Rem. 2.3] when  $p$  is not very bad for  $G$ . In this latter case, Vavilov gave a direct proof of the finiteness of the number of  $B_L$ -orbits on  $P_u$  in [40, § 4].

## 5. Proof of Theorem 1.1.

Clearly, we may assume that  $G$  is simple (over its center) and that  $A$  is a maximal closed connected abelian subgroup of  $B_u$  normalized by  $B$ . We have compiled all possibilities for such  $A$  in Table 1 in Section 3

above. Since  $A$  is  $B$ -invariant, the normalizer of  $A$  in  $G$  is parabolic in  $G$ . Throughout this section, we write  $P = LP_u$  for  $N_G(A)$  with standard Levi subgroup  $L$ .

**5.1.** — By inspection of the list in Theorem 4.2, one checks that in each of the cases in Table 1 where  $A$  is generated by a single root subgroup relative to a *simple root*  $A$  is contained in the unipotent radical of a parabolic subgroup  $Q = MQ_u$  of  $G$  whose Levi subgroup  $M$  is spherical in  $G$ . Hence, by Lemma 4.3 and Theorem 4.1,  $B_M$  and thus  $B$  act on  $Q_u$  and thus on  $A$  with a finite number of orbits. In fact, in each of these events  $Q = P$  and  $A = Q_u$ . In particular, this covers all instances for  $A_r$ ,  $C_r$ , and the first entries for  $B_r$ ,  $D_r$ ,  $E_6$ , and  $E_7$ . Also, if  $G$  is of type  $B_r$  and  $A$  is generated by the root subgroups relative to  $\beta_r$  and  $\gamma_r$ , then  $A$  is (properly) contained in the unipotent radical of the maximal parabolic subgroup  $Q$  of  $G$  corresponding to  $\Sigma \setminus \{\sigma_r\}$ . Here  $N_G(A)$  is of semisimple corank 1 in  $Q$ . This corresponds to the third case listed in Theorem 4.2 which equally applies when  $p = 2$  and then,  $Q_u$  itself is abelian. This leads to the third  $B_r$  entry in this table.

**5.2.** — Now we turn to the remaining cases in Table 1. Here we cannot appeal directly to the results from Section 4. However, a construction from [1] allows us to apply these results in part. Throughout this paragraph, suppose that  $p$  is not a very bad prime for  $G$ . In each of the cases we are concerned with  $A = C^s P_u$  for some  $s \geq 1$ . For each  $i \in \mathbb{N}$  define

$$V_i := C^{s+i-1} P_u / C^{s+i} P_u.$$

Because  $A$  is abelian, we may regard each  $V_i$  as a subgroup of  $A$ . Let

$$t := \ell(P_u) - s.$$

Then  $V_t = Z(P_u)$  is the last term in the descending central series of  $P_u$ . By the commutator relations for  $G$ , each root in  $\Psi(V_i)$  is of  $P$ -level  $s + i$ . Let  $S_i^1, S_i^2, \dots$  be the different shapes among roots of  $P$ -level  $s + i$ . For each shape  $S_i^n$  there is a unique root  $\alpha_i^n$  in  $\Psi(V_i)$  of minimal height of that shape. Let  $V_i^n$  be the product of all the root subgroups of shape  $S_i^n$ . Then each  $V_i^n$  naturally is an  $L$ -module of lowest weight  $\alpha_i^n$  and as  $L$ -modules  $V_i \cong V_i^1 \oplus V_i^2 \dots$ , cf. [1]. In particular,  $V_i^n$  is the product of all root subgroups of shape  $S_i^n$  and  $V_i$  is the product of all root subgroups of level  $s + i$ . Denote by  $\Psi_i$  the set of all roots whose shapes are integral multiples of all the occurring shapes  $S_i^n$

of level  $s + i$ . This is a closed semisimple subsystem of  $\Psi$ , and it contains  $\Psi(L)$  (the set of roots of shape 0) as a subsystem, since its elements are all multiples of the various  $S_i^n$ 's modulo the integral span of  $\Psi(L)$ . Thus the positive simple system  $\Sigma_i$  for  $\Psi_i$  consists of  $\Sigma(L)$  and the  $\alpha_i^n$ 's for  $n \geq 1$ . Since  $2S_i^n$  is not the shape of any root in  $\Psi$ , the union of  $\pm\Psi(V_i)$  and  $\Psi(L)$  equals  $\Psi_i$ . Therefore, if  $G_i$  is the connected reductive subgroup of  $G$  corresponding to  $\Psi_i$  (*i.e.*  $\Psi(G_i) = \Psi_i$ ) and containing the maximal torus  $T$ , then  $P_i = LV_i$  is the standard parabolic subgroup of  $G_i$  corresponding to  $\Sigma(L)$  with unipotent radical  $V_i$ . The number of simple components of  $G'_i$  equals the number of different shapes  $S_i^n$  of level  $s + i$ . Since  $V_i$  is abelian,  $L$  is a spherical Levi subgroup of  $G_i$  by Remark 4.4. Thus, the previous results applied to  $P_i = LV_i$  in  $G_i$  yield that  $B_L$  has a finite number of orbits on  $V_i$  for each  $i = 1, \dots, t$ . Therefore, since  $A = V_1 \cdots V_t$ , we are able to conclude our desired finiteness statement, once we have proved that

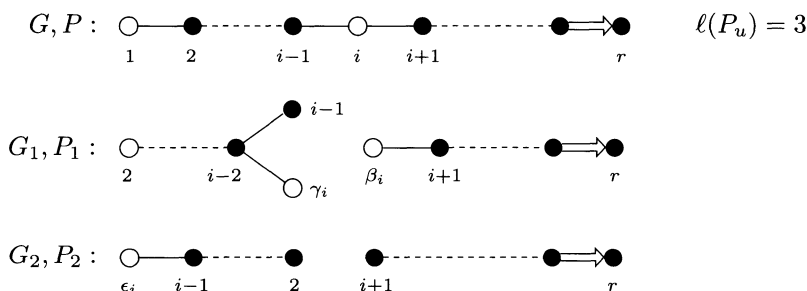
$$(\dagger) \quad \left\{ \begin{array}{l} \text{there are only finitely many } B\text{-orbits passing through each} \\ \text{coset of the form } vV_i \cdots V_t, \text{ where } v \text{ is in } V_{i-1} \setminus \{e\}, \text{ for} \\ 1 < i \leq t. \end{array} \right.$$

As  $B = B_L P_u$ , one method to establish this is to (possibly) first replace  $v$  by a suitable  $B_L$ -conjugate  $v'$  of  $v$  and then to show that each element in  $v'V_i \cdots V_t$  is already conjugate to  $v'$  under  $P_u$ . Since  $P_u$  is connected and unipotent, it has a finite number of orbits on  $v'V_i \cdots V_t$  precisely if this coset is a single  $P_u$ -orbit [33]. Sometimes another way to establish  $(\dagger)$  is more convenient. Since  $B = TB_u$ , we first aim to show that an element of  $vV_i \cdots V_t$  is  $B_u$ -conjugate to an element  $x$  which is supported by at most rank  $G$  linearly independent roots. Then  $(\dagger)$  follows, as each of the coefficients in the root elements of  $x$  can be scaled to 1 using the action of  $T$  in this event. We combine these techniques with inductive arguments below.

**5.3.** — We first attend to the remaining classical occurrences. According to 5.1 the only ones left here are the second and fourth entries for  $B_r$  as well as the last two for  $D_r$ . Here we use the notation from Section 3.

**5.3.1.** — Let  $G$  be of type  $B_r$  and let  $A$  be the normal closure in  $B$  of  $U_{\beta_i}$  and  $U_{\gamma_i}$ , where  $3 \leq i \leq r$  and suppose that  $p \neq 2$ . Then  $A = C^1 P_u$ ,  $t = 2$ , and  $\Sigma(L)$  consists of all simple roots but  $\sigma_1$  and  $\sigma_i$ , *i.e.*,  $L'$  is of type  $A_{i-2} B_{r-i}$ . If  $i = r$ , then  $L$  is of type  $A_{r-2}$ . But this case was already

discussed in 5.1. There are precisely two different  $P$ -shapes of roots of  $P$ -level 2 and the unique roots of minimal height of these two shapes are  $\beta_i$  and  $\gamma_i$  respectively. Note that  $G_1$  is of type  $D_{i-1}B_{r-i+1}$  and  $P_1 = LV_1$  is the parabolic subgroup of  $G_1$  of semisimple corank 2 corresponding to  $\beta_i$  and  $\gamma_i$ . Written additively,  $V_1$  is the sum of the alternating square of the natural module of the  $A_{i-2}$  component of  $L'$  (denoted here by  $V_1^2$ ) and of the natural module for the  $B_{r-i}$  factor ( $V_1^1$ ). The unique root of minimal height of  $P$ -level 3 is  $\epsilon_i := \beta_r + \delta_i$ , and  $G_2$  is of type  $A_{i-1}B_{r-i}$  with simple positive system  $\Sigma(L) \cup \{\epsilon_i\}$ . Here  $V_2$  is the natural module for the  $A_{i-2}$  component of  $L'$  with the second component of  $L$  acting passively. We summarize this information in the following figure. Here we indicate  $\Sigma(L)$  by coloring the corresponding nodes in the diagram of  $G$ , likewise for  $G_1$  and  $G_2$ .



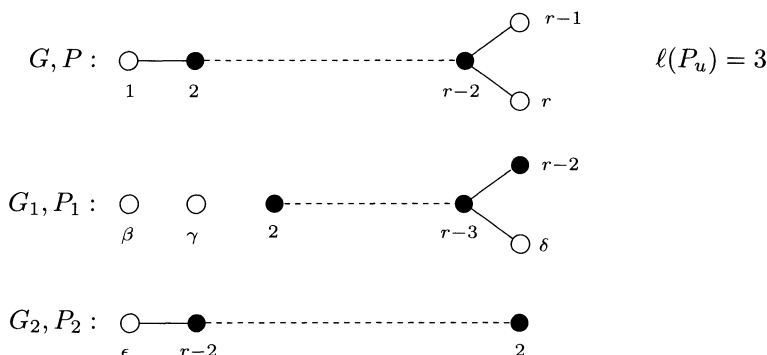
Thus  $A = V_1V_2$  and  $B_L$  has a finite number of orbits on each  $V_i$  by 5.2. Let  $v = v^1v^2$  be in  $V_1 \setminus \{e\}$ , where  $v^n \in V_1^n$  for  $n = 1, 2$ . If  $v^1 = e$ , then  $vV_2 = v^2V_2$  is contained in the unipotent radical of the parabolic subgroup of  $G$  corresponding to  $\Sigma \setminus \{\sigma_r\}$  (as every root in  $\text{supp } v^2V_2$  has coefficient 2 at  $\sigma_r$ ). If  $v^2 = e$ , then  $vV_2 = v^1V_2$  is contained in the unipotent radical of the parabolic subgroup of  $G$  corresponding to  $\Sigma \setminus \{\sigma_1\}$  (as every root in  $\text{supp } v^1V_2$  has coefficient 1 at  $\sigma_1$ ). In each one of these cases, the desired finiteness statement follows from 5.1. Now we may suppose that  $v^1 \neq e \neq v^2$ . But then every element in  $vV_2$  is  $P_u$ -conjugate to  $v$ : Let  $x = v^1v^2x'$  be in  $vV_2$  with  $x' \in V_2$  and let  $\tau$  be a root of minimal height in  $\text{supp } v^2$ . Then for each root  $\nu$  in  $\Psi(V_2)$  there is a unique root  $\mu$  in  $\Psi(P_u)$  such that  $\tau + \mu = \nu$ . By induction on height we can thus remove  $x'$  completely using the action of root elements in  $U_\mu$ . Since  $\sigma_1$  is a summand of  $\mu$ , any such operation fixes  $v^1$ , and thus  $x$  is indeed  $P_u$ -conjugate to  $v$  in this event. Thus  $(\dagger)$  is fulfilled. This completes the argument for the second  $B_r$  entry.

Next we address the fourth entry for  $B_r$ . So here  $p = 2$  and  $A$  is the normal closure in  $B$  of  $U_{\beta_i}$ ,  $U_{\delta_i}$ , and  $U_\eta$ , where  $2 \leq i \leq r - 1$ . Recall that here  $\Psi(A)$  is no longer an ideal in  $\Psi(B)$ . Let  $x$  be in  $A$ . If there is no  $\beta_j$  in  $\text{supp } x$  for  $i \leq j < r$ , then  $x$  is contained in the unipotent radical of the maximal parabolic subgroup corresponding to  $\Sigma \setminus \{\sigma_r\}$  (as then every root in  $\text{supp } x$  has a non-zero coefficient at  $\sigma_r$ ). By 5.1 there is a finite number of  $B$ -orbits in  $A$  of this nature. So we may suppose that there is a  $\beta_j$  in  $\text{supp } x$  for some  $i \leq j < r$ . Without loss, we may suppose that  $\beta_j$  is the unique such root of minimal height in  $\text{supp } x$ . Any of the remaining roots in  $\text{supp } x$  with coefficient 1 at  $\sigma_1$ , which is either  $\beta_r$  or long, can then be removed from  $\text{supp } x$  by applying suitable root elements from  $B_u$  to  $x$ . Thus we may suppose that we have an orbit representative of the form  $x = U_{\beta_j}(\xi)x'$  ( $\xi \in k^*$ ), where all roots in  $\text{supp } x'$  have coefficient 0 at  $\sigma_1$  and a non-zero coefficient at  $\sigma_r$ . Let  $H$  be the simple subgroup of  $G$  of type  $B_{r-1}$  defined by the simple system  $\Sigma' = \Sigma \setminus \{\sigma_1\}$ . Then  $x'$  belongs to the unipotent radical of the maximal parabolic subgroup  $Q$  of  $H$  corresponding to  $\Sigma' \setminus \{\sigma_r\}$ . By Theorem 4.2 the Levi subgroup  $M$  of  $Q$  is spherical in  $H$  and thus  $x'$  belongs to a finite set of orbits in  $A$  of the Borel subgroup  $B_M$  of  $M$ . By the nature of the root system  $\Psi(A)$  and since  $p = 2$ , any two elements in  $Q_u \cap A$  which are  $(B_M)_u$ -conjugate are conjugate by an element of  $(B_M)_u$  which fixes the factor  $U_{\beta_j}(\xi)$  of  $x$ . Since the maximal torus in  $B_M$  normalizes  $U_{\beta_j}$ , its action on the factor  $U_{\beta_j}(\xi)$  of  $x$  merely may result in a different coefficient  $\zeta \in k^*$ . Hence we may assume that  $x'$  belongs to a finite set of  $B_M$ -orbit representatives in  $Q_u \cap A$ . Note that each root in  $\text{supp } x'$  is orthogonal to  $\beta_j$ . Thus we may use the 1-dimensional torus  $S$  associated to  $\beta_j$  to scale the coefficient  $\zeta$  to equal 1. Ultimately, there is only a finite number of  $B$ -orbits on  $A$  in this instance as well.

5.3.2. — Next we treat the cases left for type  $D_r$  ( $r \geq 4$ ). The finiteness result for the first two entries follows from 5.1. The argument for the third entry in  $D_r$  is completely analogous to the second case for  $B_r$ . We leave the details to the reader.

In the last entry for  $D_r$  in our Table  $A$  is the normal closure in  $B$  of  $U_\beta$ ,  $U_\gamma$ , and  $U_\delta$ . Here we have  $A = C^1 P_u$  and  $\Sigma(L) = \Sigma \setminus \{\sigma_1, \sigma_{r-1}, \sigma_r\}$ , *i.e.*,  $L'$  is of type  $A_{r-3}$ . There are three different  $P$ -shapes of roots of  $P$ -level 2 and the unique roots of minimal height of these shapes are precisely  $\beta$ ,  $\gamma$ , and  $\delta$ , respectively. Note that  $G_1$  is of type  $A_1 A_1 D_{r-2}$  and  $P_1 = LV_1$  is the parabolic subgroup of  $G_1$  of semisimple corank 3 corresponding to  $\beta$ ,  $\gamma$ , and  $\delta$ . Written additively,  $V_1$  is the sum of two copies of the trivial module  $k$ ,

one for each  $A_1$  component (these correspond to  $\beta$  and  $\gamma$ , respectively, with  $L$  acting by scalars), and the alternating square of the natural module of  $L' = A_{r-3}$ . The unique root of minimal height of  $P$ -level 3 is  $\epsilon := \beta_{r-3} + \delta$  and  $G_2$  is of type  $A_{r-2}$  with simple positive system  $\Sigma(L) \cup \{\epsilon\}$ . Here  $V_2$  is the natural module for  $L'$ . We collect this data in our next figure.



So,  $A = V_1 V_2$  and by 5.2 there is a finite number of  $B_L$ -orbits on each  $V_i$ . Let  $v$  be in  $V_1 \setminus \{e\}$ . If  $\beta$  and  $\gamma$  are not in  $\text{supp } v$ , then  $vV_2$  is contained in the unipotent radical of the parabolic subgroup of  $G$  corresponding to  $\Sigma \setminus \{\sigma_r\}$  (as then every root in  $\text{supp } vV_2$  has coefficient 1 at  $\sigma_r$ ), and it follows from 5.1 that there is a finite number of  $B$ -orbits passing through  $vV_2$  in this instance. On the other hand if either  $\beta$  or  $\gamma$  is in  $\text{supp } v$ , then each element in  $vV_2$  is  $P_u$ -conjugate to  $v$ : Let  $vx'$  be in  $vV_2$  with  $x' \in V_2$ . Without loss, suppose that  $\beta \in \text{supp } v$ . Then for each root  $\nu$  in  $\Psi(V_2)$  there is a unique root  $\mu$  in  $\Psi(P_u)$  such that  $\beta + \mu = \nu$ . By induction on height we can thus remove  $x'$  completely using the action of root elements in  $U_\mu$ . Observe that, since  $\beta$  has coefficient 0 at  $\sigma_r$ , each root  $\mu$  involved has coefficient 1 at  $\sigma_r$ . Thus each operation by a root element from  $U_\mu$  on  $vx'$  fixes the other factor of  $v$  (as it consists of root elements whose support involve  $\sigma_r$  as a summand), and so  $x$  is  $P_u$ -conjugate to  $v$  in this case. Thus  $(\dagger)$  is satisfied.

We have now established Theorem 1.1 in all classical instances.

**5.4.** — We now address the bulk of the exceptional cases, the ones not covered by 5.1. For that purpose we consider the following condition: suppose that  $t = 2$  (i.e.,  $A = V_1 V_2$ ):

$$(\diamond) \quad \left\{ \begin{array}{l} \text{Suppose that for every pair of roots } (\beta, \gamma) \text{ in } \Psi(V_1) \times \Psi(V_2) \\ \text{the difference } \gamma - \beta \text{ is again a root in } \Psi \text{ (i.e., in } \Psi(P_u)) \text{ and} \\ \text{moreover, if this is the case, then we further assume that} \\ \text{for any two roots } \beta_1 \text{ and } \beta_2 \text{ in } \Psi(V_1) \text{ of the same height,} \\ \gamma - \beta_1 + \beta_2 \text{ is not a root for any } \gamma \in \Psi(V_2). \end{array} \right.$$

5.4.1. — If  $(\diamond)$  is satisfied and  $\text{char } k$  is not very bad for  $G$ , then every  $B$ -orbit passing through an element of  $A = V_1 V_2$  with nontrivial support in  $V_1$  already has a representative in  $V_1$ . The second condition in  $(\diamond)$  ensures that every  $V_2$ -factor can be removed using a succession of conjugations by suitable root elements in  $P_u$  arguing by induction on the height of the roots in  $\Psi(V_2)$ . Then, since  $B_L$  has a finite number of orbits on  $V_1$  and on  $V_2$ , there is a finite number of  $B$ -orbits on all of  $A = V_1 V_2$ . The advantage of  $(\diamond)$  is that it is a purely combinatorial condition.

One checks that in the exceptional cases with  $t = 2$  the conditions in  $(\diamond)$  are satisfied precisely in the second  $E_6$  and  $E_7$  instances, as well as in the first entries for  $E_8$ ,  $F_4$ , and  $G_2$ . Thus, our finiteness result follows in these instances provided  $p \neq 2$  if  $G$  is of type  $F_4$  or  $p \neq 3$  for  $G_2$ . Note that for  $G_2$  only the prime 3 leads to an obstruction here. But in the first  $G_2$  entry in Table 1 we require that  $p \neq 3$ .

5.4.2. — We proceed with the remaining entries for  $E_6$ . By 5.1 and 5.4.1 the finiteness result holds for the cases from the first two entries. Recall that we set  $P = LP_u = N_G(A)$ . In the third case  $\Sigma(L) = \Sigma \setminus \{\sigma_1, \sigma_5\}$ ; in particular,  $L$  is of type  $A_1 A_3$ ,  $A = C^1 P_u$ , and  $t = 2$ . Let  $\alpha_1^1 = {}^{11110}_0$  and  $\alpha_1^2 = {}^{01221}_1$ . These are the unique roots of different  $P$ -shapes and minimal height of  $P$ -level 2. Note that  $V_1^1$  is the tensor product of the natural modules for the simple factors of  $L'$  and  $V_1^2 = k$ , while  $V_2$  is the dual of the natural module for the  $A_3$ -component of  $L'$ . Thus  $\dim V_1 = 9$  and  $\dim V_2 = 4$ . Let  $v$  be in  $V_1 \setminus \{e\}$ . We consider the set of  $B$ -orbits passing through  $vV_2$ . Let  $N$  be the intersection of  $A$  with the unipotent radical of the parabolic subgroup corresponding to  $\Sigma \setminus \{\sigma_1\}$ . Then  $A = U_{\alpha_1^2} N$  and  $B$  acts on  $N$  with a finite number of orbits, by 5.1. Thus we may assume that  $\alpha_1^2$  is in  $\text{supp } v$ . But if  $\alpha_1^2 \in \text{supp } v$ , then every element in  $vV_2$  is  $P_u$ -conjugate to  $v$ : Let  $vx'$  be in  $vV_2$  with  $x' \in V_2$ . One checks that for each root  $\nu$  in  $\Psi(V_2)$  there is a unique root  $\mu$  in  $\Psi(P_u)$  such that  $\alpha_1^2 + \mu = \nu$ . By induction on height we can thus remove  $x'$  completely using the action of root elements in  $U_\mu$ . Observe that, since  $\alpha_1^2$  has coefficient 0 at  $\sigma_1$ , each root  $\mu$  involved has coefficient 1 at  $\sigma_1$ . Thus each operation by a root

element from  $U_\mu$  on  $vx'$  fixes the other factor of  $v$  (as it consists of root elements whose support involve  $\sigma_1$ ), and so  $x$  is  $P_u$ -conjugate to  $v$  in this case, as claimed. Whence  $(\dagger)$  is fulfilled.

Since the subgroup  $A$  from the fourth case for  $E_6$  is conjugate to the one from the third entry by the graph automorphism of  $G$ , the result follows readily from the previous discussion by duality.

For the final entry for  $E_6$  we have  $\Sigma(L) = \{\sigma_2, \sigma_3, \sigma_5\}$ ; in particular,  $L$  is of type  $A_1^3$ ,  $A = C^2P_u$ , and  $t = 3$ . The three “generating roots” for  $A$  are precisely the unique ones of minimal height and distinct  $P$ -shapes of  $P$ -level 3. Observe that  $V_1$  is the direct sum of three copies of natural modules for the three  $A_1$ -factors of  $L'$ . Thus  $\dim V_1 = 6$ ,  $\dim V_2 = 4$ , and  $\dim V_3 = 2$ . Let  $v$  be in  $V_1$  and write  $v = v^1v^2v^3$ , where  $v^n \in V_1^n$  for  $n = 1, 2, 3$ . Let  $N$  be the intersection of  $A$  and the abelian subgroup studied in the second case. Then  $A = U_{\alpha_1}U_{\alpha_1+\sigma_2}N$  and  $B$  acts on  $N$  with a finite number of orbits by 5.4.1. Thus we may assume that  $\text{supp } v^1$  consists of just one root, *i.e.*, either  $\alpha_1^1$ , or  $\alpha_1^1 + \sigma_2$ . Furthermore, we may also suppose that both  $v^2 \neq e$  and  $v^3 \neq e$ , as otherwise  $vV_2V_3$  is contained in the unipotent radical of the maximal parabolic subgroup corresponding to  $\Sigma \setminus \{\sigma_1\}$ , respectively  $\Sigma \setminus \{\sigma_6\}$ , and thus again, there is a finite number of  $B$ -orbits passing through  $vV_2V_3$  by 5.1 in this event. One checks that under these assumptions every element in  $vV_2V_3$  is conjugate to  $v$  under  $P_u$ : Let  $x = v^1v^2v^3x'$  be in  $vV_2V_3$  with  $x' \in V_2V_3$ . Since each of the modules  $V_1^n$  is the natural representation for one of the  $A_1$ -components in  $L'$ , the support of each  $v^n$  consists of at most two roots for each  $n = 1, 2, 3$ . While fixing the factor  $v^2v^3$  all roots in  $\text{supp } x'$  except  $\varrho$  or  $\varrho - \sigma_2$  can be removed acting suitably on  $v^1$ . The factor  $v^2v^3$  is fixed by this procedure, as each of the roots involved in the conjugation has coefficient 0 at  $\sigma_1$  and at  $\sigma_6$ . Furthermore, the single remaining root element in  $x'$  can be removed acting on  $v^2$  while fixing  $v^1v^3$  (as the conjugating root has  $\sigma_1$  as a summand). Hence we have established  $(\dagger)$ .

5.4.3. — The desired finiteness result for the first two cases for  $E_7$  was established in 5.1 and 5.4.1. We now treat the 5 remaining ones in a similar inductive manner as for  $E_6$ . For  $A$  in the third entry  $\Sigma(L) = \Sigma \setminus \{\sigma_5\}$ , so  $L$  is of type  $A_2A_4$ , where  $A = C^1P_u$ , and  $t = 2$ . Here  $V_1$  is the tensor product of the natural modules of the simple components of  $L'$  and  $V_2$  is the dual of the natural module of the  $A_4$ -factor of  $L'$ . Thus  $\dim V_1 = 15$  and  $\dim V_2 = 5$ . Suppose that  $v$  is in  $V_1 \setminus \{e\}$ . We may assume that there

is at least one root in  $\text{supp } v$  with coefficient 0 at  $\sigma_7$  (there are 5 such roots), as else  $vV_2$  is in the abelian subgroup treated in the first case. In fact, we may suppose that there is precisely one such root in  $\text{supp } v$ , the one of minimal height. The other ones can be removed using suitable root group elements from  $(B_L)_u$ . Moreover, we may suppose that there is a root in  $\text{supp } v$  whose coefficient at  $\sigma_3$  is 1, (there are six such roots), as else  $vV_2$  is contained in the subgroup considered in the second case. With these constraints one checks that either every element in  $vV_2$  is conjugate to  $v$  under  $B_u$ , or else the support of an orbit representative consists either of  ${}^{012210}_1$  and  $\varrho$ , or of  ${}^{112210}_1$  and  $\varrho - \sigma_1$ . Then, using the action of  $T$ , we can scale the coefficients of the associated root elements to equal 1 in each event. Whence the condition in (†) is satisfied.

For  $A$  from the fourth entry for  $E_7$  we have  $\Sigma(L) = \Sigma \setminus \{\sigma_3, \sigma_6\}$ , so  $L$  is of type  $A_1^2 A_3$ ,  $A = C^2 P_u$ , and  $t = 3$ . Observe that the two roots defining  $A$  are the unique ones of minimal height of  $P$ -level 3 and different  $P$ -shapes, so we denote them by  $\alpha_1^1$  and  $\alpha_1^2$  respectively. Let  $v$  be in  $V_1$  with  $v = v^1 v^2$ , where  $v^n \in V_i^n$  for  $n = 1, 2$ . Let  $N$  be the intersection of  $A$  and the abelian subgroup studied in the second  $E_7$  entry. Then  $A = U_{\alpha_1^1} U_{\alpha_1^1 + \sigma_1} N$  and  $B$  acts on  $N$  with a finite number of orbits by 5.4.1. So we may suppose that  $v^1 \neq e$ . More specifically, we may assume that  $\text{supp } v^1$  consists of either  $\alpha_1^1$ , or  $\alpha_1^1 + \sigma_1$ . Moreover, we may also assume that there is one root in  $\text{supp } v^2$  whose coefficient at  $\sigma_5$  is 1 (there are two such roots), as else  $vV_2 V_3$  is contained in the abelian subgroup from the third entry. This leads to a small list of possibilities and one checks that in each one of them any element in  $vV_2 V_3$  is conjugate to  $v$  under  $P_u$ , and so (†) is satisfied: Let  $x = vx'$  be in  $vV_2 V_3$ , with  $x' \in V_2 V_3$ . Then one checks that all roots in  $\text{supp } x'$  can be removed except possibly one (it is either  $\varrho$ , or  $\varrho - \sigma_1$ , depending on the single root in  $\text{supp } v^1$ ). However, since  $\text{supp } v$  also contains roots with coefficient 1 at  $\sigma_5$ , the single root possibly remaining in  $\text{supp } x'$  can be removed without reintroducing any new ones. Thus  $x = vx'$  is  $B_u$ -conjugate to  $v$  in this event. We have established (†) also in this case.

Next we consider the sixth case for  $E_7$ . Here  $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_7\}$ , so  $L$  is of type  $A_5$ ,  $A = C^1 P_u$ , and  $t = 2$ . Let  $v = v^1 v^2$  be in  $V_1 \setminus \{e\}$ . Note that  $V_1^2$  is the trivial module  $k$ , and thus  $\text{supp } v^2$  consists of at most one element. If  $\alpha_1^2 = {}^{123210}_2$  is not in  $\text{supp } v$ , then  $vV_2$  is contained in the unipotent radical of the maximal parabolic associated to  $\Sigma \setminus \{\sigma_7\}$  and we are done. Else, (*i.e.*, when  $v^2 \neq e$ ) every element in  $vV_2$  is  $P_u$ -conjugate to  $v$  itself: Let  $x$  be in  $vV_2$ . For every root  $\nu$  in  $\Psi(V_2)$  there is a unique root  $\mu$  in  $\Psi(P_u)$  such that  $\alpha_1^2 + \mu = \nu$ . By induction on height we can remove

the factor of  $x$  in  $V_2$  completely using the action of suitable root elements in  $U_\mu$ . Since  $\sigma_7$  must be a summand of  $\mu$ , any such operation fixes the factor  $v^1$ , and thus  $x$  is indeed  $P_u$ -conjugate to  $v$  in this instance.

Now let  $A$  be as in the fifth entry for  $E_7$ . Here  $\Sigma(L) = \Sigma \setminus \{\sigma_4, \sigma_7\}$ , so  $L$  is of type  $A_1A_2^2$ ,  $A = \mathcal{C}^2P_u$ , and  $t = 3$ . The two roots defining  $A$  are the unique ones of minimal height of  $P$ -level 3 and of different  $P$ -shapes. We denote them by  $\alpha_1^1$  and  $\alpha_1^2$  respectively. Let  $v$  be in  $V_1$  with  $v = v^1v^2$ , where  $v^n \in V_1^n$  for  $n = 1, 2$  and let  $x = vx'$  be in  $vV_2V_3$ , with  $x' \in V_2V_3$ . Let  $N$  be the intersection of  $A$  and the abelian subgroup from the sixth  $E_7$  entry just treated. Then  $A = U_{\alpha_1^1}N$  and  $B$  acts on  $N$  with a finite number of orbits by the result of the previous paragraph. So we may suppose that  $\alpha_1^2 = \text{supp } v^2$ . Observe that all but possibly three roots can then already be removed from  $\text{supp } x'$ . Furthermore, we may assume that there is precisely one root in  $\text{supp } v^1$  whose coefficient at  $\sigma_5$  is 1 (there are three such roots, and those of larger height can be removed using root elements from  $(B_L)_u$ ), as otherwise  $vV_2V_3$  is contained in the abelian subgroup treated the third case. One checks that then the two roots of smaller height of the three possible remaining ones in  $\text{supp } x'$  can be removed without introducing any new ones. If there are any other roots left in  $\text{supp } v$  then the final possible root remaining in  $\text{supp } x'$  can also be removed and consequently  $x$  is  $B_u$ -conjugate to  $v$ . If there are no further roots in  $\text{supp } v$ , then  $x$  is supported by at most three linearly independent roots. Once again this establishes  $(\dagger)$  in this instance.

Finally, we address the last  $E_7$  entry. In this case  $\Sigma(L) = \{\sigma_1, \sigma_2, \sigma_4, \sigma_6\}$ , so  $L$  is of type  $A_1^2A_2$ ,  $A = \mathcal{C}^3P_u$ , and  $t = 4$ . Let  $v$  be in  $V_1$ . Let  $V = V_2V_3V_4$ . We may suppose that  ${}^{122111}_1$  is in  $\text{supp } v$  (note  $V_1^3 = k$ ), as else  $vV$  is contained in the subgroup we studied in the third case (this also applies for  $v = e$ ). Furthermore, we may suppose that there is a root in  $\text{supp } v$  whose coefficient at  $\sigma_3$  equals 1, else  $vV$  is contained in the abelian subgroup from the second entry (there are four such roots). These two conditions together already lead to a small list of possible configurations in this event, and it turns out that then every element in  $vV$  is already  $P_u$ -conjugate to  $v$ : Let  $x = vx'$  be in  $vV$  with  $x' \in V$ . Since  ${}^{122111}_1$  is in  $\text{supp } v$ , all but possibly one of the roots in  $\text{supp } x'$  can be removed using root subgroups from  $P_u$ . The remaining root in  $\text{supp } x'$  can then be removed by acting on one of the root elements in  $v$  relative to a root with coefficient 1 at  $\sigma_3$  without introducing any new roots from  $\Psi(V)$ . Thus we have  $(\dagger)$  also in this case. This completes the discussion for  $E_7$ .

5.4.4. — For  $E_8$  the finiteness result for the first case follows from 5.4.1. We treat the remaining 7 cases again in the same inductive manner as done for  $E_6$  and  $E_7$ . In the second case  $\Sigma(L) = \Sigma \setminus \{\sigma_5\}$ , so  $L$  is of type  $A_3A_4$ ,  $A = C^2P_u$ , and  $t = 3$ . Here  $\dim V_1 = 20$ ,  $\dim V_2 = 10$ , and  $\dim V_3 = 4$ . Let  $v_1$  be in  $V_1$ . We may suppose that there is a root in  $\text{supp } v_1$  whose coefficient at  $\sigma_7$  equals 1, as else  $v_1V_2V_3$  is contained in the subgroup from the first case (also for  $v_1 = e$ ). There are ten such roots. Using the action of  $(B_L)_u$  we may suppose that there are at most two such (orthogonal) roots in  $\text{supp } v_1$ . Let  $x = v_1v_2v_3$  be in  $V_1V_2V_3$ , where  $v_i \in V_i$  for  $i = 2, 3$  and  $v_2v_3 \neq e$ . One checks that for a fixed  $\beta \in \Psi(V_1)$  there are precisely four roots  $\gamma \in \Psi(V_2)$  such that  $\gamma - \beta$  is not a root and there is precisely one root  $\delta \in \Psi(V_3)$  such that  $\delta - \beta$  is not a root in  $\Psi$ . Let  $\tau$  be a root in the support of  $v_1$  of minimal height and with coefficient 1 at  $\sigma_7$ . Thus, we may assume that all roots of the form  $\tau + \eta$  with  $\eta \in \Psi(B_L)$  have been removed from the support of  $v_1$  and that  $v_2$  consists of at most four root elements and that either  $v_3 = e$ , or  $v_3$  is a single root element (using the action of  $P_u$ ). If there are no further roots in the support of  $v_1$ , then  $x$  is supported by at most six linearly independent roots and we can use the maximal torus  $T$  in  $B$  to scale the coefficients in the remaining root elements of  $x$  to equal 1. Now suppose there are two (orthogonal) roots in  $\text{supp } v_1$  each with coefficient 1 at  $\sigma_7$ , say  $\tau$  and  $\gamma$ , and that all roots of the form  $\tau + \eta$  with  $\eta \in \Psi(B_L)$  have been removed from the support of  $v_1$ , likewise for  $\gamma$ . Then all of the 5 remaining roots but possibly one in  $\text{supp } v_2$  can be removed from  $\text{supp } v_2v_3$  without introducing any new roots. If there are no further roots in  $\text{supp } v_1$ , then  $x$  is supported by three linearly independent roots. Furthermore, if there are still any additional roots in  $\text{supp } v_1$  (necessarily orthogonal to  $\tau$ ), then (using the action of  $P_u$ ) we can remove the final root element factor from  $v_2$  without reintroducing any new root elements or ones which have already been removed. Thus, in this latter instance we are left with an orbit representative in  $V_1$ . Consequently, we have a finite number of  $B$ -orbits on all of  $A = V_1V_2V_3$ , as desired.

For  $A$  as in the third entry  $\Sigma(L) = \Sigma \setminus \{\sigma_2\}$ , so  $L$  is of type  $A_7$ ,  $A = C^1P_u$ , and  $t = 2$ . Here  $V_1$  is the alternating square of the natural module of  $A_7$ , thus  $\dim V_1 = 28$  and  $V_2$  is the 8-dimensional natural  $A_7$ -module. Let  $v$  be in  $V_1$ . We may suppose that there is at least one root in  $\text{supp } v$  whose coefficient at  $\sigma_5$  is 2, as otherwise  $vV_2$  is contained in the subgroup studied in the second case. There are just six such roots in  $\Psi(V_1)$ . We may further suppose that there are at most two such orthogonal roots in  $\text{supp } v$ . If needed, the other four can be removed using suitable

root elements from  $B_L$ . Now let  $x = vx'$  be a  $B$ -orbit representative in  $A = V_1V_2$ , where  $x' \in V_2$ . One checks that for a fixed  $\beta \in \Psi(V_1)$  there are precisely two roots  $\gamma \in \Psi(V_2)$  such that  $\gamma - \beta$  is not a root. Let  $\delta$  be a root in the support of  $v$  with coefficient 2 at  $\sigma_5$ . Thus, we may assume that all roots of the form  $\delta + \eta$  with  $\eta \in \Psi(B_L)$  have been removed from the support of  $v$  and that  $x'$  consists of at most two root elements. If there is a second root in  $\text{supp } v$  with coefficient 2 at  $\sigma_5$  (orthogonal to  $\delta$ ), then we can remove both remaining root element factors of  $x'$  and thus  $x$  is  $B_u$ -conjugate to  $v$ . So we may suppose that  $\delta$  is the single root in  $\text{supp } v$  with coefficient 2 at  $\sigma_5$ . If there are no further roots in the support of  $v$ , then  $x$  is supported by at most three linearly independent roots and we use the maximal torus  $T$  in  $B$  to scale the coefficients in the various root elements to 1 and obtain a single orbit representative. Furthermore, one checks that if there are any extra roots in  $\text{supp } v$ , apart from the ones already removed (*i.e.*, necessarily orthogonal ones to  $\delta$ ), then we can remove the factor  $x'$  by conjugating with suitable root elements from  $P_u$ , without reintroducing any root factors which have already been removed and we are left with an orbit representative in  $V_1$ . Consequently, we have a finite number of  $B$ -orbits on  $A = V_1V_2$  in each event.

Now consider the fourth case. Here  $\Sigma(L) = \Sigma \setminus \{\sigma_1, \sigma_7\}$ , so  $L$  is of type  $A_1D_5$ ,  $A = C^2P_u$ , and  $t = 3$ . Let  $v = v^1v^2$  be in  $V_1$ . Note that  $V_1^1$  is a 16-dimensional spin module for the  $D_5$ -component of  $L'$  and  $V_1^2$  is the natural module for the  $A_1$  factor. We may suppose that there is precisely one of the two roots in  $\text{supp } v^2$  with coefficient 1 at  $\sigma_7$ , as otherwise  $vV_2V_3$  is contained in the subgroup studied in the first case (also for  $v = e$ ). In addition, we may suppose that there are roots in  $\text{supp } v$  whose coefficient at  $\sigma_5$  is 2, otherwise  $vV_2V_3$  is contained in the subgroup from the second case. Let  $x = vx'$  be in  $vV_2V_3$  with  $x' \in V_2V_3$ . Acting on  $v^2$  with suitable root elements from  $P_u$  we can remove all but possibly one root from  $\text{supp } x'$ . This fixes the  $v^1$ -factor, as each of the roots involved has coefficient 0 at  $\sigma_1$ . Acting on  $v^1$  we can remove this remaining root from  $\text{supp } x'$  while fixing  $v^2$ , as any root element needed here has coefficient 1 at  $\sigma_1$  and thus commutes with  $v^2$ . Therefore, these two conditions combined imply that every element in  $vV_2V_3$  is already conjugate to  $v$  under  $P_u$  and thus  $(\dagger)$  is satisfied.

For the fifth  $E_8$  entry  $\Sigma(L) = \Sigma \setminus \{\sigma_3, \sigma_7\}$ . So  $L$  is of type  $A_1^2A_4$ ,  $A = C^3P_u$ , and  $t = 4$ . Let  $v = v^1v^2$  be in  $V_1$ . We write  $V$  for  $V_2V_3V_4$ . The two defining roots for  $A$  are the unique ones of minimal height and different  $P$ -shapes of level 4. Note that  $\dim V_1 = 14$  and  $\dim V = 17$ . We may suppose in this instance that there is precisely one of the two roots in

the support of  $v^2 \in V_1^2$  with coefficient 1 at  $\sigma_1$ , as else  $vV$  is contained in the subgroup treated in the fourth case (this also applies for the case  $v = e$ ). Let  $x = vx'$  be in  $vV$  with  $x' \in V$ . One checks that all put possibly six roots in  $\text{supp } x'$  can be removed acting on  $v^2$ . In addition, we may suppose that there is precisely one of the three roots in  $\text{supp } v^1$  with coefficient 2 at  $\sigma_5$ , as else  $vV$  is contained in the subgroup from the second case. Then we may remove up to four more roots from  $\text{supp } x'$ . If there are no other roots in  $\text{supp } x$ , then  $x$  is supported by at most four linearly independent roots, and we are done. Else,  $x$  is  $B_u$ -conjugate to  $v$ , and  $(\dagger)$  is satisfied.

If  $A$  is as in the sixth entry for  $E_8$ , then  $\Sigma(L) = \Sigma \setminus \{\sigma_4, \sigma_7\}$ . So  $L$  is of type  $A_1^2 A_2^2$ ,  $A = C^4 P_u$ , and  $t = 5$ . Write  $V$  for  $V_2 \cdots V_5$ . Let  $v = v^1 v^2$  be in  $V_1$ . The two defining roots for  $A$  are the unique ones of minimal height and different  $P$ -shapes of level 5. Note that  $\dim V_1 = 12$  and  $\dim V = 20$ . Let  $x = vx'$  be in  $vV$  with  $x' \in V$ . We may suppose that there is precisely one (of the two) roots in  $\text{supp } v^2$  with coefficient 2 at  $\sigma_3$ , as else  $vV$  is contained in the subgroup studied in case 5 (this also applies for the case  $v = e$ ). Then we can remove all but nine roots from  $\text{supp } x'$ . Furthermore, we may suppose that one of the two roots with coefficient 2 at  $\sigma_5$  is in  $\text{supp } v^1$ . Otherwise  $vV$  is contained in the subgroup from in case 2. Acting on root elements in  $v$  of the second kind we can eliminate up to another six roots in  $\text{supp } x'$ . If there are no additional roots in  $\text{supp } v$ , then  $x$  is supported by at most five linearly independent roots and we may use the action of  $T$ . Otherwise, all of the remaining roots in  $\text{supp } x'$  can be removed, so that  $x$  is conjugate to  $v$  under  $B_u$ , as desired.

For  $A$  from the seventh  $E_8$  entry  $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_6\}$ . So  $L$  is of type  $A_2 A_4$ ,  $A = C^3 P_u$ , and  $t = 4$ . Write again  $V$  for  $V_2 V_3 V_4$ . Let  $v = v^1 v^2$  be in  $V_1$ . We may suppose that there is precisely one of the three roots in  $\text{supp } v^2$  with coefficient 2 at  $\sigma_5$ , as else  $vV$  is contained in the subgroup studied in case 2 (this also applies when  $v = e$ ). Note that  $V_1^1 = k$ . Moreover, we may assume that  $\alpha_1^1 = {}^{1233321}_1$  is in  $\text{supp } v$ , as otherwise  $vV$  is contained in the maximal abelian subgroup from case 3. These conditions imply that either an element in  $vV$  is  $P_u$ -conjugate to  $v$  itself, or it belongs to a  $B_u$ -orbit with a representative  $x$  in  $vV$  whose support consists of three linearly independent roots,  $\alpha_1^1$ ,  $\varrho - (\sigma_6 + \sigma_7 + \sigma_8)$ , and one of  $\alpha_1^2$ ,  $\alpha_1^2 + \sigma_8$ , or  $\alpha_1^2 + \sigma_7 + \sigma_8$ . In each event we use the action of  $T$  to scale the coefficients of the corresponding root elements to 1 and so  $(\dagger)$  is satisfied.

Finally, we address the last  $E_8$  case. Here  $\Sigma(L) = \Sigma \setminus \{\sigma_2, \sigma_5, \sigma_7\}$  and  $L$  is of type  $A_1^2 A_3$ ,  $A = C^5 P_u$ , and  $t = 6$ . Write again  $V$  for  $V_2 \cdots V_6$ . Here

$\dim V_1 = 11$  and  $\dim V = 22$ . Let  $v = v^1 v^2 v^3$  be in  $V_1$  and as before let  $x = vx'$  be in  $vV$  with  $x' \in V$ . Note that  $V_1^2 = k$  and so  $|\text{supp } v^2| \leq 1$ . We may suppose that  $\alpha_1^2 = {}^{1232221}_2$  is in  $\text{supp } v$ , as otherwise  $vV$  is contained in the maximal abelian subgroup from case 2. This is the single root in  $\Psi(V_1)$  with coefficient 2 at  $\sigma_5$ . Acting on  $v^2$  with root elements from  $P_u$  we can remove all but possibly up to eight roots from  $\text{supp } x'$ . Also, we may suppose that there is precisely one of the two roots in  $\text{supp } v^3$  with coefficient 3 at  $\sigma_4$ , as else  $vV$  is contained in the maximal abelian subgroup from case 6. Then we can act on  $v^3$  and remove an additional four roots from  $x'$  without reintroducing any new ones. If there are no other roots in  $\text{supp } v$ , then  $x$  is supported by at most eight linearly independent roots and we are done using the action of  $T$ . Otherwise, we can use other root elements to completely eliminate  $x'$ , and thus  $x$  is conjugate under  $B_u$  to  $v$ . This completes the discussion for  $E_8$ .

5.4.5. — We proceed with the remaining events for  $F_4$ . The desired result for the first entry in Table 1 for  $F_4$  was established for  $p \neq 2$  in 5.4.1. First we consider the same instance when  $p = 2$ . As before,  $A = C^1 P_u$ , where  $P = N_G(A)$ . Although  $\langle \diamond \rangle$  is satisfied, the commutator relations in  $F_4$  are degenerate when  $p = 2$  for short root subgroups spanning a subsystem of  $\Psi$  of type  $B_2$ . Let  $v_1$  be in  $V_1 \setminus \{e\}$ . We consider the set of  $B$ -orbits passing through  $v_1 V_2$ . Since the two roots in  $\Psi(V_2)$  are long and there are no degeneracies for the commutators of long root subgroups in  $\Psi(B)$ , we can argue as above, provided there is a long root in  $\text{supp } v_1$ . Consequently, we only need to consider the case when  $v_1$  is supported entirely on short roots. Since any two short roots in  $\Psi(V_1)$  span a subsystem of Dynkin type  $A_2$  (and the structure constants in the corresponding group are  $\pm 1$ ), we may assume (possibly after using the action of  $(B_L)_u$ ) that  $v_1$  is a single short root element. Thus, any element in  $v_1 V_2 = v_1 U_{1342} U_{2342}$  is supported by at most three linearly independent roots, and we can use the action of the torus  $T$  in  $B$  to obtain a finite set of orbit representatives in this case as well.

Next we turn to the remaining  $F_4$  cases. It is advantageous to first consider a particular normal abelian subgroup of  $B$  which, although not maximal, still fits the setting of 5.2. Namely, let  $V$  be the normal closure in  $B$  of  $U_\delta$ , where  $\delta = 0122$ . Let  $P = N_G(V)$ . Then  $P = LP_u$  is the maximal parabolic subgroup of  $G$  of type  $B_3$ ,  $V = Z(P_u)$ , and  $\dim V = 7$ . We can apply the same construction as in 5.2. Let  $\Psi_1$  be the closed semisimple subsystem of  $\Psi$  formed by  $\Psi(L)$  together with  $\pm \Psi(V)$  and positive simple

system consisting of  $\Sigma(L)$  and  $\delta$ . Let  $G_1$  be the semisimple subgroup of  $G$  corresponding to  $\Psi_1$ . Then  $G_1$  is of type  $B_4$  and  $P_1 = LV$  is the maximal parabolic subgroup of  $G_1$  corresponding to the simple root  $\delta$ . Since  $V$  is abelian,  $L$  is a spherical Levi subgroup of  $G_1$  by Remark 4.4. Hence,  $B_L$  and thus  $B$  has a finite number of orbits on  $V$ . Note this is valid also for  $p = 2$ .

Now in the second  $F_4$  incident on our list  $A = U_{1221}U_{1231}V$ . Thus, by the last paragraph, it suffices to consider orbit representatives whose support contains at least one of 1221 or 1231. One easily checks that there is only a finite number of such  $B$ -orbits passing through  $A$ .

In the third case for  $F_4$  we have  $A = U_{0121}U_{1121}U_{1221}U_{1231}V$ . As above, we only need to consider elements whose support involves at least one of the first four roots shown. Keeping in mind that  $p = 2$  in this instance, one checks directly that there is only a finite number of such  $B$ -orbits in  $A$ .

The commutative groups from the last two  $F_4$  entries (both for  $p = 2$  only) share the 10-dimensional normal subgroup of  $B$  which is generated by  $U_{1121}$  and  $U_{0122}$ . Thus, by the previous result, in the fourth case we only need to show that there are finitely many  $B$ -orbits passing through elements in  $A$  whose support involves 1111. This is easily verified.

5.4.6. — Finally, the result for the second  $G_2$  case is readily checked directly.

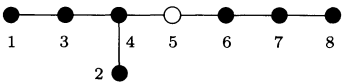
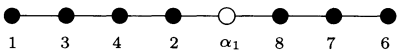
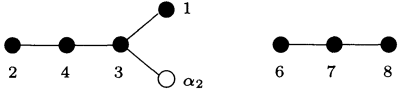
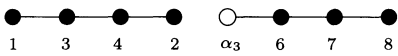
This completes the proof of Theorem 1.1.

We close this section by illustrating the construction from 5.2 once again with two examples from Table 1, this time for exceptional groups.

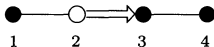
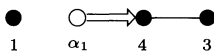
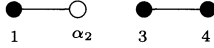
**Examples 5.5.** — Our first example is the second entry for  $E_8$  from Table 1. The Levi subgroup of  $N_G(A) = P = LP_u$  is of type  $A_3A_4$ ,  $\ell(P_u) = 5$ ,  $A = \mathcal{C}^2P_u$ , and  $t = 3$ . As shown,  $G_i$  is of type  $A_8$ ,  $A_3D_5$ , or  $A_4A_4$  for  $i = 1, 2, 3$ , respectively. As before we indicate  $\Sigma(L)$  by coloring the corresponding nodes in the diagram of  $G$  and  $G_i$ . Moreover,  $\alpha_i$ ,  $\dim V_i$ , and the various passive components of  $G_i$  are specified as well.

We also present the first case for  $F_4$ . Here the Levi subgroup of  $N_G(A)$  is of type  $A_1\tilde{A}_2$ , where  $\tilde{A}_2$  is the subsystem of  $\Psi$  spanned by the short simple roots, and  $G_1$  and  $G_2$  are of type  $A_1C_3$  and  $A_2\tilde{A}_2$ , respectively.

---

|             |                                                                                   |                                     |
|-------------|-----------------------------------------------------------------------------------|-------------------------------------|
| $G, P:$     |  | $\ell(P_u) = 5$                     |
| $G_1, P_1:$ |  | $\alpha_1 = 1233210, \dim V_1 = 20$ |
| $G_2, P_2:$ |  | $\alpha_2 = 1244321, \dim V_2 = 10$ |
| $G_3, P_3:$ |  | $\alpha_3 = 2465321, \dim V_3 = 4$  |

---

|             |                                                                                   |                                 |
|-------------|-----------------------------------------------------------------------------------|---------------------------------|
| $G, P:$     |  | $\ell(P_u) = 3$                 |
| $G_1, P_1:$ |  | $\alpha_1 = 1220, \dim V_1 = 6$ |
| $G_2, P_2:$ |  | $\alpha_2 = 1342, \dim V_2 = 2$ |

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6. Abelian ideals of Lie  $P$ .

Let  $G$ ,  $P$ , and  $A$  be as in the Introduction. If  $\text{char } k$  is zero, then the exponential mapping is a  $P$ -equivariant morphism between  $\mathfrak{a} = \text{Lie } A$  and  $A$ . If  $\text{char } k$  is a good prime for  $G$ , we can make use of Springer’s map  $\varphi: \mathcal{U} \longrightarrow \mathcal{N}$  between the unipotent variety  $\mathcal{U}$  of  $G$  and the nilpotent variety  $\mathcal{N}$  of  $\mathfrak{g}$  which is a  $G$ -equivariant bijective morphism (see [35]), and, upon “restriction” of  $\varphi$  to  $A$ , we obtain a  $P$ -equivariant bijective morphism from  $A$  onto  $\mathfrak{a}$ , *e.g.*, see [31, Thm 4.1]. Note that  $\mathfrak{a}$  is an abelian ideal in  $\mathfrak{p}$ . Consequently, we get a statement analogous to Theorem 1.1 for the adjoint action of  $P$  on  $P$ -invariant linear subspaces of  $\text{Lie } P_u$  which are abelian as subalgebras of  $\mathfrak{p}$ . Thanks to a result of Pyasetskii [27], which is also valid in positive characteristic, we also obtain a statement for the coadjoint action of  $P$  on  $\mathfrak{a}^*$  similar to Theorem 1.1.

The characteristic restrictions for the adjoint action can be removed completely in the exceptional cases in Table 1 by employing the computer

algorithm outlined in [14]. This algorithm is valid provided  $p$  is not very bad for  $G$ . The remaining cases when  $p = 2$  for  $F_4$  and  $G_2$  can be verified directly.

## 7. Abelian subalgebras of $\mathfrak{g}$ .

In [21] A.I. Mal'cev determined all abelian subalgebras of maximal dimension in each simple complex Lie algebra  $\mathfrak{g}$ , up to  $G$ -conjugacy, extending work of I. Schur [34], *i.e.*, the special case for  $\mathfrak{sl}_n(k)$ .

We give an approach to Mal'cev's result in arbitrary characteristic utilizing the information in Table 1 above. Let  $\mathfrak{s}$  be a solvable subalgebra of  $\mathfrak{g}$ . Without loss, we may assume that  $\mathfrak{s} \subset \mathfrak{b}$  (Lie-Kolchin). Let  $X$  be the Grassmann variety consisting of flags of subspaces  $\mathfrak{s}' \subset \mathfrak{b}'$  in  $\mathfrak{g}$ , where  $\dim \mathfrak{s}' = \dim \mathfrak{s}$  and  $\dim \mathfrak{b}' = \dim \mathfrak{b}$ . Then  $X$  is projective and  $G$  operates on it via the adjoint action. Let  $Y$  be the closure of the  $G$ -orbit in  $X$  of the flag  $\mathfrak{s} \subset \mathfrak{b}$ . Whence  $Y$  is complete. Observe that  $Y$  consists of flags  $\mathfrak{s}' \subset \mathfrak{b}'$  in  $\mathfrak{g}$ , where  $\mathfrak{s}'$  is a solvable subalgebra of  $\mathfrak{g}$  and  $\mathfrak{b}'$  is a Borel subalgebra of  $\mathfrak{g}$ . By Borel's fixed point theorem [2, 10.4] there is a flag  $\mathfrak{s}_0 \subset \mathfrak{b}_0$  in  $Y$  which is fixed by  $B$ . Thus  $\mathfrak{b} = \mathfrak{b}_0$  and  $\mathfrak{s}_0$  is an ideal in  $\mathfrak{b}$ . In other words, there is always an ideal of  $\mathfrak{b}$  in the closure of the  $G$ -orbit in  $\mathfrak{g}$  of any solvable subalgebra  $\mathfrak{s}$  of  $\mathfrak{b}$  (possibly of different isomorphism type, but of the same dimension as  $\mathfrak{s}$ ).

Considering the particular case when  $\mathfrak{s}$  is an abelian subalgebra of  $\mathfrak{g}$ , the maximal possible dimensions of these can be read off from Table 1 above by the aforementioned construction. Comparing the information in this table with Mal'cev's list, we observe that in all instances, with the exception of  $G_2$ , every abelian subalgebra of  $\mathfrak{g}$  of maximal dimension is itself conjugate to an abelian ideal of  $\mathfrak{b}$  under  $G$  (provided  $p$  is not very bad for  $G$ ). In  $G_2$  there are three classes of abelian subalgebras of  $\mathfrak{g}$  of maximal dimension 3 ( $p \neq 2$ ); but there is only one such class containing the abelian ideal  $\text{Lie } A$  of  $\mathfrak{b}$ , where  $A$  is as in Table 1.

Although, it is not directly related to the questions addressed in this paper, we should like to mention recent work of B. Kostant [19], extending earlier results from [18], where the family of *all* abelian ideals  $\mathfrak{a}$  of the Borel subalgebra  $\mathfrak{b}$  of a simple complex Lie algebra  $\mathfrak{g}$  plays an important rôle. Motivated by Mal'cev's work [21], Kostant constructs inequivalent irreducible  $G$ -submodules in the exterior algebra  $\wedge \mathfrak{g}$  of  $\mathfrak{g}$ , one for each abelian ideal  $\mathfrak{a}$  of  $\mathfrak{b}$  in [18], where  $\text{Lie } G = \mathfrak{g}$ . In his recent summary [19],

Kostant gives an account of D. Peterson's remarkable theorem that the number of abelian ideals in  $\mathfrak{b}$  equals  $2^r$ , where  $r = \text{rank } \mathfrak{g}$ . He determines the structure of each such ideal  $\mathfrak{a}$  in terms of a certain Cartan decomposition of  $\mathfrak{g}$ , and moreover, exhibits a close connection between the discrete series representations and the abelian ideals  $\mathfrak{a}$ .

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