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ON GRADIENTS OF FUNCTIONS DEFINABLE IN O-MINIMAL STRUCTURES

by Krzysztof KURDYKA

0. Introduction.

Many results in subanalytic or semialgebraic geometry of $\mathbb{R}^n$ hold true in a more general setting called "the theory of o-minimal structures on the real field" (see [DM]). This theory has presented a strong interest since 1991 when A. Wilkie [W1] proved that a natural extension of the family of semialgebraic sets containing the exponential function ($(\mathbb{R}, \exp)$-definable sets) is an o-minimal structure. A similar extension of subanalytic sets ($(\mathbb{R}_{\mathrm{an}}, \exp)$-definable sets) was then treated by L. van den Dries, A. Macintyre, D. Marker in [DMM] (geometric proofs of these facts were found recently by J-M. Lion and J.-P. Rolin [LR1], [LR2]). Another type of o-minimal structure ($(\mathbb{R}^K_{\mathrm{an}})$-definable sets) was obtained by C. Miller [Mi], by adding to subanalytic sets all functions $x \to x^r$, $r \in K$, where K is a subfield of $\mathbb{R}$. We give a list and examples of o-minimal structures in section 1. An extension of semialgebraic and subanalytic geometry was also undertaken by M. Shiota [S1], [S2].

Theorem 1 (Section 2), the first main result of this paper, is an o-minimal generalization of the famous Łojasiewicz inequality $\|\mathbf{grad}\, f\| \geq |f|^\alpha$ with $\alpha < 1$, where f is an analytic function in a neighborhood of $a \in \mathbb{R}^n$, $f(a) = 0$. We prove that if f is a differentiable function in a

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bounded domain, definable in some o-minimal structure, then there exists a C^1 function Ψ in one variable such that $\|\mathbf{grad}\,\Psi \circ f\| \geq c > 0$. It is rather surprising that the result holds also for infinitely flat functions. Theorem 1 implies that the set of asymptotic critical values of f is finite (Proposition 2). We recall in the beginning of the section the already known o-minimal version of another Łojasiewicz inequality for continuous definable functions on a compact set.

The main result of Section 3 is Theorem 2 which states: if U is an open, bounded subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is a C^1 function definable in some o-minimal structure, then all trajectories of $-\mathbf{grad}\, f$ (*i.e.* solutions of the equation $\dot{x} = -\mathbf{grad}\, f$) have their length bounded by a constant independent of the trajectory. The function f may be unbounded and may not have a continuous extension on $\overline{U}$. We prove also, that for a non negative definable g, the flow of $-\mathbf{grad}\, g$ defines a deformation retraction onto $g^{-1}(0)$. Some applications of this result in the real analytic case can be found in [Si], [Sj]. We finish the paper by a discussion of Thom's Gradient Conjecture for o-minimal structures.

In Section 1 we gather basic facts on o-minimal structures. To make the paper self-contained and accessible for a wider audience we add a proof of Lemma 2 (on definable functions in one variable). We give also an elementary proof (suggested by C. Miller and J-M. Lion) of the curve selection lemma, the crucial tool in the proof of Theorem 1.

General references of various facts, when not specified, will be as follows: for semialgebraic geometry – [BCR], for subanalytic geometry – [BM] or [Ł4], for o-minimal structures – [DM].

In this paper we take the gradient with respect to the canonical euclidian metric in $\mathbb{R}^n$.

1. o-minimal structures on the real field.

DEFINITION 1. — *Let $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$. We say that the collection $\mathcal{M}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$ if:*

(1) *each $\mathcal{M}_n$ is closed under finite set-theoretical operations;*

(2) *if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$;*

(3) *let* $A \in \mathcal{M}_{n+m}$ *and* $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ *be projection on the first* n *coordinates, then* $\pi(A) \in \mathcal{M}_n$*;*

(4) *let* $f, g_1, ..., g_k \in \mathbb{Q}[X_1, ..., X_n]$*, then* $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, ..., g_k(x) > 0\} \in \mathcal{M}_n$*;*

(5) $\mathcal{M}_1$ *consists of all finite unions of open intervals and points.*

For a fixed o-minimal structure $\mathcal{M}$ *on* $(\mathbb{R}, +, \cdot)$ *we say that* A *is an* $\mathcal{M}$*-set if* $A \in \mathcal{M}_n$ *for some* $n \in \mathbb{N}$*. We say that a function* $f : A \longrightarrow \mathbb{R}^m$*, where* $A \subset \mathbb{R}^n$*, is an* $\mathcal{M}$*-function if its graph is an* $\mathcal{M}$*-set.*

Axiom (5) will be called the o-minimality of $\mathcal{M}$.

Examples. — We give below a list of o-minimal structures on $(\mathbb{R}, +, \cdot)$ (see also [DM] for detailed definitions and comparisons between the above examples) with examples of functions definable in those o-minimal structures:

(1) Semialgebraic sets (by Tarski-Seidenberg); $f(x, y) = \sqrt{x^4 + y^4}$.

(2) Global subanalytic sets (by Gabrielov); $f(x, y) = \dfrac{y}{\sin x}, x \in (0, \pi)$.

(3) $(\mathbb{R}, \exp)$-definable sets (by Wilkie); $f(x, y) = x^2 \exp\Big(-\dfrac{y^2}{x^4 + y^2}\Big) \ln x$.

(4) $(\mathbb{R}_{an}, \exp)$-definable sets (by van den Dries, Macintyre, Marker); $f(x, y) = x^{\sqrt{2}} \ln(\sin y)$, $x > 0, y \in (0, \pi)$.

(5) $(\mathbb{R}_{an}^{\mathbb{R}})$-definable sets (by Miller); $f(x, y) = x^{\sqrt{2}} \exp\Big(\dfrac{x}{y}\Big)$, $0 < x < y < 1$.

Recently another example of an o-minimal structure was found by van den Dries and Speissegger [DS] which is larger than $\mathbb{R}_{an}^{\mathbb{R}}$ but polynomially bounded (*i.e* any definable function in one variable is bounded by a polynomial at infinity). Finally we mention a result of Wilkie [W2] in which he gives a general method for construction of o-minimal structures; this method can be applied to Pfaffian functions.

In the rest of this paper $\mathcal{M}$ **will denote some fixed, but arbitrary, o-minimal structure on** $(\mathbb{R}, +, \cdot)$. We will give now several elementary properties of $\mathcal{M}$-sets and $\mathcal{M}$-functions.

Remark 1. — Let E be an $\mathcal{M}$-set in $\mathbb{R}^{n+1}$. Axioms (1)–(4) imply that the sets

$$\{x \in \mathbb{R}^n : \exists x_{n+1}\ (x, x_{n+1}) \in E\} \quad \text{and} \quad \{x \in \mathbb{R}^n : \forall x_{n+1}\ (x, x_{n+1}) \in E\}$$

are $\mathcal{M}$-sets. Actually the first set is the image of E by projection, the second is the complement of the image of the complement of E by projection.

Remark 2. — The sum, product, inverse, composition of $\mathcal{M}$-functions is again an $\mathcal{M}$-function. Also the image and inverse image of an $\mathcal{M}$-set by an $\mathcal{M}$-function are again $\mathcal{M}$-sets. Proofs of these facts are quite standard applications of Remark 1 and axioms (1)–(4) and actually the same as in the semialgebraic case (see *e.g.* [BCR]).

LEMMA 1. — *Let $f : A \longrightarrow \mathbb{R}$ be an $\mathcal{M}$-function such that $f(x) \geq 0$ for all $x \in A$. Let $G : A \longrightarrow \mathbb{R}^m$ be an $\mathcal{M}$-mapping and define a function $\varphi : G(A) \longrightarrow \mathbb{R}$ by*

$$\varphi(y) = \inf_{x \in G^{-1}(y)} f(x).$$

Then φ is an $\mathcal{M}$-function.

Proof. — Write a formula for the graph of the function φ and apply Remark 1.

COROLLARY 1. — *Let A be an $\mathcal{M}$-set in $\mathbb{R}^n$. Then the distance function $d_A : \mathbb{R}^n \to \mathbb{R}$ is an $\mathcal{M}$-function, where $d_A(x) = \inf_{y \in A} |x - y|$.*

COROLLARY 2. — *Let A be an $\mathcal{M}$-set in $\mathbb{R}^n$. Then $\overline{A}$ and* Int A *are $\mathcal{M}$-sets.*

Proof. — Actually by Corollary 1 we know that d_A is an $\mathcal{M}$-function, hence $\overline{A} = d_A(0)^{-1}$ is an $\mathcal{M}$-set. To prove that the interior of A is an $\mathcal{M}$-set we use the fact that by axiom (1) the complement of an $\mathcal{M}$-set is an $\mathcal{M}$-set.

LEMMA 2 (Monotonicity Theorem). — *Let $f : (a, b) \longrightarrow \mathbb{R}$ be an $\mathcal{M}$-function. Then there exist real numbers $a = a_0 < a_1 < \ldots < a_k = b$ such that f is continuously differentiable on each interval (a_i, a_{i+1}). Moreover f' is an $\mathcal{M}$-function and the function f is strictly monotone or constant on every interval (a_i, a_{i+1}).*

Proof (Due essentially to van den Dries [vD]). — We may assume that the set $f((a, b))$ is infinite. First we prove that $D(f)$, the set of discontinuity points of f, is finite.

Writing the definition of continuity of a function at a point and using Remark 1 we deduce that $D(f)$ is an $\mathcal{M}$-set in $\mathbb{R}$, hence by o-minimality, it is enough to prove that f is continuous at some point of (a,b). Since the set $f((a,b))$ is an infinite $\mathcal{M}$-set it contains an open interval. Thus by induction we can construct a descending sequence of intervals $[\alpha_n, \beta_n] \subset (a,b)$ such that $\alpha_n < \alpha_{n+1}$, $\beta_{n+1} < \beta_n$, $\beta_n - \alpha_n < 1/n$ and $f([\alpha_n, \beta_n])$ is contained in an interval of length smaller than $1/n$. Clearly f is continuous at the point $\bigcap_{n \in \mathbb{N}} [\alpha_n, \beta_n]$. So we have proved that the complement of $D(f)$ is dense in (a,b), hence $D(f)$ is finite.

We can assume now that f is continuous on (a,b). To prove differentiability observe first that by o-minimality we have:

OBSERVATION. — *For each $x \in (a,b)$ and each $c \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that $f(t) \geq f(x) + c(t-x)$ for all $t \in (x, x+\varepsilon)$ or $f(t) \leq f(x) + c(t-x)$ for all $t \in (x, x+\varepsilon)$.*

Let us write $f'_-(x) = \lim_{t \nearrow 0} \frac{1}{t}(f(x+t) - f(x))$ for $x \in (a,b]$ and $f'_+(x) = \lim_{t \searrow 0} \frac{1}{t}(f(x+t) - f(x))$ for $x \in [a,b)$. Note that f'_+ and f'_- are $\mathcal{M}$-functions, by Remark 1. From the above observation it is not difficult to obtain the following consequences:

i) for each $x \in (a,b)$ the values of $f'_-(x)$ and $f'_+(x)$ are well defined (possibly equal to $+\infty$ or $-\infty$),

ii) for each $x \in (a,b)$ there exists y arbitrary close to x , $y > x$ such that $f'_+(y) \leq f'_+(x)$, $f'_-(y) \leq f'_+(x)$ or $f'_+(y) \geq f'_+(x)$, $f'_-(y) \geq f'_+(x)$.

Clearly the sets

$$\{x \in (a,b); f'_+(x) = +\infty\},\ \{x \in (a,b); f'_+(x) = -\infty\}$$

are $\mathcal{M}$-sets, hence are finite unions of open intervals and points. By ii) these sets are finite. So we can assume that f'_+ and f'_- take values in $\mathbb{R}$. Since f'_+ and f'_- are $\mathcal{M}$-functions we may also assume that these functions are continuous on (a,b). It follows easily now from ii) that $f'_+ = f'_-$ on (a,b), but this means that f is C^1 on (a,b).

We proved also that f' is an $\mathcal{M}$-function, hence the claim on monotonicity follows from the fact that $\{f' = 0\}$ is an $\mathcal{M}$-set and so is a finite union of points and open intervals.

Writing the definition of partial derivatives and using Remark 1 we obtain:

LEMMA 3. — *Let $f : U \longrightarrow \mathbb{R}^k$ be a differentiable $\mathcal{M}$-function, where U is open in $\mathbb{R}^n$. Then $\partial f/\partial x_j$, $j = 1, \dots, n$ are $\mathcal{M}$-functions, and hence* **grad** f *is an $\mathcal{M}$-mapping.*

PROPOSITION 1 (Curve Selection Lemma). — *Let A be an $\mathcal{M}$-set in $\mathbb{R}^n$ and suppose that $a \in \overline{A \setminus \{a\}}$. Then there exists an $\mathcal{M}$-function $\gamma : [0, \varepsilon) \longrightarrow \mathbb{R}^n$ which is C^1 on $[0, \varepsilon)$ and such that*

$$a = \gamma(0) \quad \text{and} \quad \gamma((0, \varepsilon)) \subset A \setminus \{a\}.$$

Proof. — The key point is to construct a "definable" selection operator e, which assigns to each nonempty set $A \in \mathcal{M}_n$ an element $e(A) \in A$. Let $n = 1$. Then $e(A)$ is the smallest element of A if A has one. Otherwise, let $a := \inf A$ and let $b \in \mathbb{R} \cup \{+\infty\}$ be maximal such that $(a, b) \subseteq A$. If $a, b \in \mathbb{R}$, then $e(A) := (a + b)/2$. If $a \in \mathbb{R}$ and $b = +\infty$, then $e(A) := a + 1$. If $a = -\infty$ and $b \in \mathbb{R}$, then $e(A) := b - 1$. If $a = -\infty$ and $b = +\infty$ (*i.e.*, $A = \mathbb{R}$), then $e(A) := 0$. Assume $e(A)$ has been defined for all nonempty $A \in \mathcal{M}_n$. Let $B \in \mathcal{M}_{n+1}$ be nonempty, and let A be its image in $\mathbb{R}^n$ under the projection map $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Put $a := e(A)$. Then $e(B) := (a, e(B_a))$ where $B_a := \{r \in \mathbb{R} : (a, r) \in B\}$.

This selection operator e has several applications, and Curve Selection is only one of them: let $A \in \mathcal{M}_n$ and $a \in \overline{A \setminus \{a\}}$. By o-minimality the set $\{|a - x| : x \in A\} \in \mathcal{M}_1$ contains an interval $(0, \epsilon)$, $\epsilon > 0$. For $0 < t < \epsilon$, let $\gamma(t) := e(\{x \in A : |a - x| = t\})$. It is routine to check that $\gamma : (0, \epsilon) \to A$ belongs to $\mathcal{M}$. By the monotonicity theorem γ is C^1 after suitable shrinking of ϵ. After composition on the right with a sufficiently flat (at 0) function in $\mathcal{M}$ (e.g. the inverse of the bigest component of γ) we can further arrange that γ extends to a C^1-function on $[0, \epsilon)$.

2. Łojasiewicz inequalities for o-minimal structures.

We begin this section recalling an already well-known generalization of the Łojasiewicz inequality for continuous $\mathcal{M}$-functions on a compact set. This result was observed by T. Loi [Lo] for $(\mathbb{R}, \exp)$-definable sets (actually his version is more precise than the theorem stated below); M. Shiota [S1], [S2] and L. van den Dries and C. Miller [DM] also noticed this fact.

THEOREM 0. — *Let K be a compact subset of $\mathbb{R}^n$ and let $f, g : K \longrightarrow \mathbb{R}$ be two continuous $\mathcal{M}$-functions. If $f^{-1}(0) \subset g^{-1}(0)$, then there* *exists a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class C^1, such that for any $x \in K$ we have*

$$|f(x)| \geq \sigma(g(x)).$$

The idea of the proof goes back to the original argument of Łojasiewicz (see [Ł2] , [KŁZ]). Let $\Sigma \subset \mathbb{R}^2$ be the image of K by the mapping $K \ni u \to (g(u), f(u)) = (x, y)$. Clearly Σ is an $\mathcal{M}$-set; moreover it is compact and $\Sigma \cap \{y = 0\} = \{(0,0)\}$. It is not difficult to find (by Lemma 2) a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class C^1, such that $\Sigma \subset \{y \geq \sigma(x),\ x \geq 0\}$. It is proved in [DM] that for each $k \in \mathbb{N}$ one can find σ of class C^k.

We state now the main result of this section. Recall that $\mathcal{M}$ is any fixed o-minimal structure on $(\mathbb{R}, +, \cdot)$.

THEOREM 1. — *Let $f : U \longrightarrow \mathbb{R}$ be a differentiable $\mathcal{M}$-function, where U is an open and bounded subset of $\mathbb{R}^n$. Suppose that $f(x) > 0$ for all $x \in U$. Then there exists $c > 0$, $\rho > 0$ and a strictly increasing positive $\mathcal{M}$-function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ of class C^1, such that*

$$\|\mathbf{grad}\,(\Psi \circ f)(x)\| \ \geq c,$$

for each $x \in U$, $f(x) \in (0, \rho)$.

The proof is given in the end of the section. We shall see now that in the subanalytic case our Theorem 1 is equivalent to the classical Łojasiewicz inequality for gradients of analytic functions (see [Ł1], [Ł2], [BM]). We state this result in the form generalized in [KP]:

THEOREM (ŁI). — *Let $f : \Omega \longrightarrow \mathbb{R}$ be a subanalytic function which is differentiable in $\Omega \setminus f^{-1}(0)$, where Ω is an open bounded subset of $\mathbb{R}^n$. Then there exist $C > 0, \rho > 0$ and $0 \leq \alpha < 1$ such that:*

$$\|\mathbf{grad}\, f(x)\| \ \geq C|f(x)|^\alpha,$$

for each $x \in \Omega$ such that $|f(x)| \in (0, \rho)$. If in addition $\lim_{x \to a} f(x) = 0$ for some $a \in \overline{\Omega}$ (which holds in the classical case, where f is analytic and $a \in \Omega$, $f(a) = 0$), then the above inequality holds for each $x \in \Omega \setminus f^{-1}(0)$ close to a.

To see that in the subanalytic case (ŁI) $\Rightarrow$ Theorem 1 it is enough to put $\Psi(t) = t^{1-\alpha}$. To prove the converse in the subanalytic case, recall first that every subanalytic function in one variable is actually semianalytic (see [Ł2], [KŁZ]). Hence Ψ has the Puiseux expansion of the form $\Psi(t) = \sum_{\nu=0}^{\infty} a_\nu t^{\frac{\nu}{k}}$. Thus, for t small enough we have $|\Psi'(t)| \leq D\, t^{\frac{1}{k}-1}$ for some $D > 0$. The last inequality and Theorem 1 yield

$$\|\mathbf{grad}\, f(x)\| = \frac{\|\mathbf{grad}\,(\Psi \circ f)(x)\|}{|\Psi' f(x)|} \geq \frac{c}{D}|f(x)|^{1-\frac{1}{k}}.$$

Remark. — The above argument and Theorem 1 imply that (ŁI) holds in any polynomially bounded o-minimal structure on $(\mathbb{R}, +, \cdot)$.

We discuss now a consequence of Theorem 1. Let $f : U \longrightarrow \mathbb{R}$ be a differentiable function, where U is an open subset of $\mathbb{R}^n$. We shall say that $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$ is an *asymptotic critical value* of f if there exists a sequence $x_n \in U$ such that

$$f(x_n) \to \lambda \text{ and } \mathbf{grad}\, f(x_n) \to 0.$$

Clearly any "true" critical value of f (*i.e* $\lambda = f(x)$ and $\mathbf{grad}\, f(x) = 0$, for some $x \in U$) is also an asymptotic critical value. Notice that this notion depends heavily on the domain U, in particular on whether U is bounded or not.

Suppose now that U is bounded and that our f is an $\mathcal{M}$-function, where $\mathcal{M}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$. Let λ be an asymptotic critical value of f. It follows immediately from Theorem 1 that f has no asymptotic critical values in $(\lambda - \rho, \lambda) \cup (\lambda, \lambda + \rho)$ for some $\rho > 0$. But on the other hand the set of all asymptotic critical values of f is an $\mathcal{M}$-subset of $\mathbb{R}$, so it must be finite. Thus we have proved:

PROPOSITION 2. — *If U is bounded and f is an $\mathcal{M}$-function, then the set of all asymptotic critical values of f is finite.*

It is easily seen that $-\infty$ and $+\infty$ cannot be an asymptotic critical value of an $\mathcal{M}$-function defined in a bounded set. As the following example shows the assumption of boundness on U is necessary.

Example. — The function $f(x, y) = \dfrac{x}{y}$ on $U = \{y > 0\} \subset \mathbb{R}^2$, being semialgebraic, belongs to any o-minimal structure on $(\mathbb{R}, +, \cdot)$. But clearly any $\lambda \in \mathbb{R}$ is an asymptotic critical value of f.

Proof of Theorem 1. — It follows from Lemma 3 that $U \ni x \mapsto \|\mathbf{grad}\, f(x)\|$ is an $\mathcal{M}$-function. We may suppose that $f^{-1}(t) \neq \emptyset$ for any small enough $t > 0$, since otherwise, by o-minimality, the theorem is trivial. Hence the function

$$\varphi(t) = \inf\{\|\mathbf{grad}\, f(x)\| : x \in f^{-1}(t)\}$$

is well-defined in some interval $(0, \varepsilon)$. By Lemma 1, φ is an $\mathcal{M}$-function.

CLAIM. — *There exists $\varepsilon' > 0$ such that $\varphi(t) > 0$ for any $t \in (0, \varepsilon')$.*

Assume that this is not the case and put

$$\Sigma = \{x \in U : \|\mathbf{grad}\, f(x)\| < (f(x))^2\}.$$

Clearly Σ is an $\mathcal{M}$-set. Let $f|_\Sigma$ denote the graph of f restricted to Σ. If the claim doesn't hold, then there exists a sequence of positive numbers $t_n \to 0$ such that $\varphi(t_n) = 0$ for all $n \in \mathbb{N}$. Let $x_n \in \Sigma$ be a sequence such that $f(x_n) = t_n$, in other words $(x_n, t_n) \in f|_\Sigma$. Let b be an accumulation point of $\{x_n\}$, then $(b, 0)$ belongs to the closure of the set $(f|_\Sigma \setminus \{(b, 0)\})$. By the curve selection lemma (Proposition 1) we have an $\mathcal{M}$-function (arc) $\tilde{\gamma} : (-\delta, \delta) \to \mathbb{R}^n \times \mathbb{R}$ of class C^1, such that $\tilde{\gamma}(0) = (b, 0)$, and $\tilde{\gamma}(0, \delta) \subset f|_\Sigma$. Write $\tilde{\gamma}(s) = (\gamma(s), f \circ \gamma(s))$, where $\gamma(s) \in \Sigma \subset \mathbb{R}^n$. Let $h(s) = f \circ \gamma(s)$ for $s \in (0, \delta)$, then clearly $\lim_{s \to 0} h(s) = 0 = \lim_{s \to 0} h'(s)$, since $\gamma(s) \in \Sigma$. Of course h and h' are $\mathcal{M}$-functions, so by Lemma 2 we may suppose that h and h' are monotone; actually they must be strictly increasing. Thus we have

$$0 < h'(s) \leq A(h(s))^2, \quad \text{for} \quad s \in (0, \delta),$$

where A is a bound for $\|\gamma'(s)\|$. But by the Mean Value Theorem we have $h(s) \leq s\, h'(s)$, because h' is increasing. Finally, we get $0 < h'(s) \leq As^2(h'(s))^2$ for any $s \in (0, \delta)$, which is impossible since $\lim_{s \to 0} h'(s) = 0$.

So we have proved that $\varphi(t) > 0$ for all $t \in (0, \varepsilon)$, provided that $\varepsilon > 0$ is small enough. We define now:

$$\Delta = \{x \in U \setminus f^{-1}(0) : f(x) < \varepsilon,\ \|\mathbf{grad}\, f(x)\| \leq 2\varphi(f(x))\}.$$

Observe that Δ is also an $\mathcal{M}$-set and moreover $\Delta \cap f^{-1}(t) \neq \emptyset$ for every $t \in (0, \varepsilon)$. Hence as before there exists $d \in \overline{U}$ such that $(d, 0) \in \overline{f|_\Delta \setminus \{(d, 0)\}}$. Applying again the curve selection lemma to $f|_\Delta$ at the point $(d, 0)$ we obtain an $\mathcal{M}$-function (arc) $\tilde{\eta} : (-\delta, \delta) \to \mathbb{R}^n$ of class C^1, such that $\tilde{\eta}(0) = (d, 0)$, and $\tilde{\eta}(0, \delta) \subset f|_\Delta$. Write as before $\tilde{\eta}(s) = (\eta(s), f \circ \eta(s))$, where $\eta(s) \in \Delta \subset \mathbb{R}^n$. Let $g(s) = f \circ \eta(s)$ for $s \in (0, \delta)$, then clearly $\lim_{s \to 0} g(s) = 0$ and $g(s) > 0$ for each $s \in (0, \delta)$. It follows from Lemma 2 that for $\delta' > 0$ small enough the function $g : (0, \delta') \longrightarrow \mathbb{R}$ is a diffeomorphism onto $(0, \rho)$, for some $\rho > 0$. We put

$$\Psi(t) = g^{-1}(t) \quad \text{for} \quad t \in (0, \rho).$$

We shall check now the inequality claimed in Theorem 1. Let B be some bound for $\|\eta'(s)\|$ in $(0,\delta')$. Take any $x \in U$ such that $t = f(x) \in (0,\rho)$, and write $s = \Psi(t) = g^{-1}(t)$. Then we have

$$\|\mathbf{grad}\,\Psi \circ f(x)\| = \Psi'(f(x))\|\mathbf{grad}\, f(x)\|$$
$$\geq \Psi'(t)\frac{1}{2}\|\mathbf{grad}\, f(\eta(s))\| \geq \frac{\Psi'(t)}{2B}(f \circ \eta)'(s) = \frac{1}{2B} = c,$$

since $\|\mathbf{grad}\, f(\eta(s))\|\,\|\eta'(s)\| \geq \langle \mathbf{grad}\, f(\eta(s)), \eta'(s)\rangle = (f \circ \eta)'(s)$ and $B \geq \|\eta'(s)\|$. Theorem 1 follows.

3. Trajectories of gradients of $\mathcal{M}$-functions.

Let $f : U \longrightarrow \mathbb{R}$ be a C^1 function, where U is an open subset of $\mathbb{R}^n$. We shall consider a vector field,

$$U \ni x \mapsto -\mathbf{grad}\, f(x) \in \mathbb{R}^n.$$

Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$. We shall say that $\gamma : (\alpha, \beta) \to U$ is a *trajectory of the vector field* $-\mathbf{grad}\, f$ if it is a maximal differentiable curve verifying $\gamma'(t) = -\mathbf{grad}\, f(\gamma(s))$. Actually we shall consider γ as an equivalence class of all curves obtained from γ by a strictly increasing C^1 reparametrization. Observe that if ψ is an increasing C^1 diffeomorphism between two intervals in $\mathbb{R}$, then the trajectories of $-\mathbf{grad}\,\psi \circ f$ and those of $-\mathbf{grad}\, f$ are the same.

Let $a, b \in \gamma$. We denote by $|\gamma(a,b)|$ the length of γ between a and b.

Łojasiewicz derived (see [Ł1], [Ł3]) from (ŁI) that all trajectories of $-\mathbf{grad}\, f$ are of finite length, when f is analytic in a neighborhood of a compact $\overline{U}$. We have:

THEOREM 2. — *Let $f : U \longrightarrow \mathbb{R}$ be a function of class C^1, where U is an open and bounded subset of $\mathbb{R}^n$. Suppose that f is an $\mathcal{M}$-function, for some o-minimal structure $\mathcal{M}$.*

a) *Then there exists $A > 0$ such that all trajectories of $-\mathbf{grad}\, f$ have length bounded by A.*

b) *More precisely, there exists $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous strictly increasing $\mathcal{M}$-function, with $\lim_{t\to 0}\sigma(t) = 0$, such that if γ is a trajectory of $-\mathbf{grad}\, f$ and $a, b \in \gamma$, then*

$$|\gamma(a,b)| \leq \sigma(|f(b) - f(a)|).$$

Proof of theorem 2. — Taking, if necessary the composition $\psi \circ f$, where $\psi(t) = \dfrac{t}{\sqrt{1+t^2}}$, we may suppose that f is bounded; more exactly that the image of f lies in $(-1,1)$. We consider again the $\mathcal{M}$-function $\varphi : (-1,1) \to \mathbb{R}$ defined by

$$\varphi(t) = \inf\{\|\mathbf{grad}\, f(x)\| : x \in f^{-1}(t)\},$$

when $f^{-1}(t) \neq \emptyset$, and $\varphi(t) = 1$ when $f^{-1}(t) = \emptyset$. Let Σ be the set of all asymptotic critical values of f. Observe that $\lambda \in \Sigma$ if $\varphi(\lambda) = 0$, or $\lim_{t \nearrow \lambda} \varphi(t) = 0$, or $\lim_{t \searrow \lambda} \varphi(t) = 0$.

Let $I \subset (-1,1)$ be an open interval. Assume that φ is bounded from below in I by some $c > 0$. Let γ be a trajectory of $-\mathbf{grad}\, f$ and $a, b \in \gamma$. Suppose that the part of γ lying between a and b is contained in $f^{-1}(I)$. We parametrise γ by arc-length (*i.e* $\|\gamma'(s)\| = 1$), so by the Mean Value Theorem we have that $|f \circ \gamma(\beta) - f \circ \gamma(\alpha)| \geq c\,|\beta - \alpha|$, in other words

$$|\gamma(a,b)| \leq \frac{1}{c}|f(b) - f(b)|.$$

This observation explains the idea of the proof. By a partition $-1 = t_0 < t_1 < \ldots < t_k = 1$ we shall decompose $(-1,1)$ in such a way that φ is strictly monotone on (t_i, t_{i+1}). Moreover we shall distinguish between two disjoint types of intervals, namely

(1) there exists $c_i > 0$ such that $\varphi(t) \geq c_i$ on (t_i, t_{i+1}) (we write $i \in I_1$ in this case), or

(2) one of t_i, t_{i+1} is an asymptotic critical value of f, hence by Theorem 1, there exist $c_i > 0$ and $\Psi_i : (t_i, t_{i+1}) \to \mathbb{R}$ a strictly increasing, bounded C^1 function such that,

$$\|\mathbf{grad}\, (\Psi_i \circ f)(x)\| \geq c_i$$

for all $x \in f^{-1}(t_i, t_{i+1})$ (we write $i \in I_2$ in this case).

Take now any trajectory γ of $-\mathbf{grad}\, f$, and let $\gamma_i = \gamma \cap f^{-1}(t_i, t_{i+1})$. We denote by $|\gamma|$ (resp. $|\gamma_i|$) the length of γ (resp. γ_i). Clearly $|\gamma_i| \leq \dfrac{1}{c_i}|t_i - t_{i+1}|$ if $i \in I_1$. Extending by continuity, we may suppose that each Ψ_i is defined also at t_i and t_{i+1}. Hence for $i \in I_2$ we have $|\gamma_i| \leq \dfrac{1}{c_i}|\Psi_i(t_i) - \Psi_i(t_{i+1})|$, since the trajectories of $-\mathbf{grad}\,(\Psi_i \circ f)$ and $-\mathbf{grad}\, f$ are the same in $f^{-1}(t_i, t_{i+1})$. Finally, we can write

$$|\gamma| = \sum_{i=0}^{k-1} |\gamma_i| \leq \sum_{i \in I_1} \frac{1}{c_i}|t_i - t_{i+1}| + \sum_{i \in I_2} \frac{1}{c_i}|\Psi_i(t_i) - \Psi_i(t_{i+1})| = A,$$

which proves part a) of Theorem 2.

We are now going to construct the function σ of part b). For $i \in I_2$ we put

$$\sigma_i(r) = \frac{1}{c_i} \sup\{|\Psi_i(p) - \Psi_i(q)| \,:\, p, q \in (t_i, t_{i+1}),\, r = p - q\},$$

and $\sigma_i(r) = \dfrac{r}{c_i}$ for $i \in I_1$. Extend each σ_i to a continuous strictly increasing $\mathcal{M}$-function on $\mathbb{R}$. It is easily seen that $\sigma = \sup \sigma_i$ satisfies b) of Theorem 2.

We finish this section by a short discussion of some consequences of Theorem 2, which extend and generalize those known in the real analytic (compact) setting.

Observe that if $\gamma : (\alpha, \beta) \to U$ is a trajectory then $x_0 = \lim_{s \to \beta} \gamma(s)$ exists, and in general x_0 belongs to $\overline{U}$. Notice that if $x_0 \in U$, then x_0 is a critical point of f. Let us take E a closed $\mathcal{M}$-subset in an open set U; by 4.22 of [DM], E is the zero set of an $\mathcal{M}$-function $f : U \to \mathbb{R}$ of class C^2. Let $g = f^2$. We want to show that the flow of $-\mathbf{grad}\, g$ defines a strong deformation retraction of a neighborhood of E onto E. This is actually a new result even in the subanalytic case since the retraction is global and E is not necessarily compact. By Proposition 2, taking a neighborhood of E, we may suppose that 0 is the only asymptotic critical value of g in U. Clearly the set

$$V = \{x \in U :\ \mathrm{dist}(x, \partial U) < \sigma(g(x))\}$$

is an $\mathcal{M}$-set, it is an open neighborhood of E. For each $x \in V$ we denote by $\gamma_x : (\alpha_x, \beta_x) \to U$ the trajectory passing through x. It is clearly unique if $g(x) \neq 0$ and constant (hence unique) if $g(x) = 0$. Put $R(x) = \lim_{s \to \beta_x} \gamma_x(s)$, and observe that $R(x) \in E$. We have:

PROPOSITION 3. — *There exists an open neighborhood V_1 of E such that $R : V_1 \longrightarrow E$ is a strong deformation retraction.*

Proof. — First we shall prove that R is continuous. Take $x_0 \in V$ and Ω_0 a neighborhood of $R(x_0)$. Let $x_1 \notin E$ be close to $R(x_0)$ so that there is (by Theorem 2 b)) a neighborhood Ω_1 of x_1 with the following property: any trajectory passing through Ω_1 has its limit in Ω_0. By continuity of the flow of $-\mathbf{grad}\, g$ there exists a neighborhood G of x_0 such that any trajectory passing by G must cross Ω_1. So we have $R(G) \subset \Omega_0$, which proves the continuity of R.

Let γ be the trajectory passing through x. Let γ_x be the part of γ between x and the limit $R(x)$. Assume that $\gamma_x : [0, \beta_x] \to U$ is parametrized by arc-length; moreover that $\gamma_x(0) = x$, and $\gamma_x(\beta_x) = R(x)$. Clearly β_x is the length of γ_x. Notice that the argument in the proof of continuity of R yields that the function $V \ni x \to \beta_x$ is continuous. Let V_1 be the set of all $x \in V$ such that γ_x lies in V. We define a homotopy $F : [0,1] \times V_1 \longrightarrow V_1$ as follows: $F_t(x) = \gamma_x(t\beta_x)$.

In general the retraction R is not an $\mathcal{M}$-mapping. Take $g(x,y) = (x^2-y^3)^2$; it was observed by Hu [Hu] that the retraction R is not hoelderian (at $(0,0)$) in this case, hence it cannot be subanalytic. Observe also that, in general, the set V_1 is not an $\mathcal{M}$-set. It would be interesting to prove that actually R belongs to some larger o-minimal structure. Even a weaker problem is open (also in the subanalytic case):

CONJECTURE (F). — *Let γ be a trajectory of $-\mathbf{grad}\, f$, where f is an $\mathcal{M}$-function of class C^1, and let H be any $\mathcal{M}$-subset. Then $\gamma \cap H$ has a finite number of connected components.*

This is connected with the Gradient Conjecture of R. Thom, proved recently in [KM]. R. Thom asked whether for an analytic function f every trajectory γ of $-\mathbf{grad}\, f$ has a tangent at the limit point (*i.e.* whether $\lim_{s\to\beta_x} \dfrac{\gamma(s) - R(x)}{|\gamma(s) - R(x)|}$ exists). We can of course ask the same question for a trajectory of the gradient of any $\mathcal{M}$-function of class C^1.

It is easily seen that (F) implies that $\lim_{s\to\beta_x} \dfrac{\gamma'(s)}{|\gamma'(s)|}$ exists, thus that the tangent to γ at the limit point exists.

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