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ON EQUIVARIANT HARMONIC MAPS DEFINED ON A LORENTZ MANIFOLD

by MA LI

1. Introduction.

It is interesting to study harmonic maps from a Lorentz manifold into a Riemannian manifold. In this case, the harmonic map equation is a Hyperbolic system of second order. In this paper, we look for equivariant harmonic maps defined on a specific Lorentz manifold ; namely, the Lorentz manifold $M = M_0 \times R$ with the space-time metric

$$ds^2 = dt^2 - S^2(t) d\sigma^2$$

where $(M_0, d\sigma^2)$ be the symmetric space for a compact Lie group G with a bi-invariant Riemannian metric $d\sigma^2$ and $S(t)$ is a smooth positive function defined on R . The target manifold is a compact Riemannian manifold (N, h) admitting an isometric group action of G . This kind of problem is called a σ -model in Physics literature and one may see [G] and [EL] for further datum. Without loss of generality, we may assume that N is a submanifold of some Euclidean space R^k by Nash's isometrical imbedding theorem, so we may think of G as $\subset SO(k)$ with its Lie algebra $LG \subset so(k)$ the Lie algebra of $SO(k)$ whose elements are skew-matrices.

By definition, a smooth map u from M to N is called a harmonic map if it is a critical point of the following action integral

$$\begin{aligned} E_I(u) &= \int_{M_0 \times I} (Tr_{ds^2} u^* h) S^n(t) d\mu dt \\ &= \int_I \int_{M_0} (|\partial_t u|^2 - S^{-2}(t) |\nabla_0 u|^2) S^n(t) d\mu dt \end{aligned}$$

for every interval $I = [a, b] \subset R$ among all maps of its class, here $n = \dim M_0$, $d\mu$ is the invariant measure of $(M_0, d\sigma)$, $|\cdot|$ is the usual norm induced by R^k and ∇_0 is the covariant derivative induced by $d\sigma^2$ on M_0 .

We will prove the following

THEOREM. — *Let M and N be the manifold defined above. Suppose $S(t)$ is a smooth positive periodic function of period 2π , then, there exist infinitely many G -equivariant harmonic maps which are of period 2π in t from M to N .*

By equivariant, we mean that the map $u: M \rightarrow N$ satisfies

$$u(g \cdot m, t) = g \cdot u(m, t)$$

for every $g \in G$ and $(m, t) \in M_0 \times R$. We denote the set of equivariant maps \mathcal{M} and it is non-empty by our assumptions on M and N . Select a basis $\{e_j\}_{j=1}^n$ (note $n = \dim M_0 = \dim_{\mathbb{R}} G$) of the Lie group G and let $\{A_j\}_{j=1}^n$ denote the corresponding basis of its Lie algebra. Fix $m \in M_0$ and write $x(t) = u(m, t)$. Because u is an equivariant map, $u(\exp(sA_j)m, t) = \exp(sA_j)u(m, t)$. Differentiating it w.r.t. s at $s = 0$ we get that $\nabla_0 u(m, t)(A_j) = A_j u(m, t)$ (matrix multiplication in R^k). From this and the invariance of the metric $d\sigma^2$, the action integral $E_I(\cdot)$ for the G -equivariant map u becomes

$$\begin{aligned} E_I(u) &= \int_{M_0} d\mu \int_I (|u_t(m, t)|^2 - S^{-2}(t) \sum_{j=1}^n |A_j(u(m, t))|^2) S^n(t) dt \\ &= \text{Vol}(M_0) F_I(x), \end{aligned}$$

where the last integral factor $F_I(x)$ will be written as $F(x)$ when $I = S^1$.

It will be shown by the minimax principle that there exist infinitely many critical points of $F(\cdot)$ just like closed geodesics in N . But here we should mention that it is conceptually different from the closed geodesic case because the Euler-Lagrange equation for our $F(\cdot)$ is a non-autonomous one (see Lemma 2 below).

2. Some well-known facts.

Since the A_j is a skew-symmetric matrix, there exists a non-negative symmetric matrix A such that

$$\sum_{j=1}^n A_j^2 = - \sum_{j=1}^n A_j A_j^* = -A^2.$$

So

$$(1) \quad F(x) = \int_0^{2\pi} (|x'(t)|^2 - S^{-2}(t)|Ax(t)|^2) S^n(t) dt.$$

Think of $M_0 \times S^1$ as a Riemannian manifold with the metric $dt^2 + S^2(t) d\sigma^2$, we may define a Hilbert manifold $H = W^{1,2}(M_0 \times S^1, N)$ for l large enough. Now, H admits an isometric group action $(u, g) \rightarrow g^{-1} \cdot u \cdot g$ of G . Applying the theorem in page 23 of R. S. Palais [P2] to F on H and to the fixed point set of the map $u \rightarrow g^{-1} \cdot u \cdot g$, we find

LEMMA 1. — *If $u \in \mathcal{M}$, then u is harmonic if and only if $x(t) = u(m, t)$ is a critical point for $F_I(x)$ for all intervals $I \subset R$.*

Let \mathcal{O} be an open uniform tubular neighborhood of N in R^k such that the $P: \mathcal{O} \rightarrow N$ given by $P(y) =$ the nearest point in N to y , is a smooth fibration.

LEMMA 2. — *The Euler-Lagrange equations for an equivariant harmonic map from M to N are*

$$(2) \quad S^{-n}(t)(S^n(t)x')' - D^2P(x', x') + S^{-2}(t)A^2x = 0,$$

which is a non-autonomous system except if $S(t) = \text{const.}$

Proof. — Suppose x is the critical point of $F(\cdot)$ which corresponds to the equivariant harmonic map we consider. For $\eta \in W^{1,2}(S^1, R^k)$, if $\varepsilon > 0$ is small enough, we have that $P(x(\cdot) + s\eta(\cdot))$ is a smooth curve in $W^{1,2}(S^1, N) := \{y \in W^{1,2}(S^1, R^k); y(t) \in N\}$ passing through x for $s \in (-\varepsilon, \varepsilon)$. Hence

$$\begin{aligned} 0 &= 2^{-1} d/ds|_{s=0} F(P(x + s\eta)) \\ &= \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \int_{S^1} |DP_{x+s\eta} \cdot (x'(t) + s\eta'(t))|^2 S^n(t) dt \\ &\quad - \int_{S^1} |AP(x(t) + s\eta(t))|^2 S^{n-2}(t) dt \\ &= \int_{S^1} (\langle DP_x \cdot x'(t), D^2P_x(x'(t), \eta(t)) + DP_x \cdot \eta'(t) \rangle) S^n(t) dt \\ &\quad - \int_{S^1} \langle A^2x(t), \eta(t) \rangle S^{n-2}(t) dt \end{aligned}$$

$$\begin{aligned}
&= \int_{S^1} \langle x'(t), \eta'(t) \rangle S^n(t) dt \\
&+ \int_{S^1} (\langle D^2 P_x(x'(t), x'(t)) - S^{-2}(t) A^2 x(t), \eta(t) \rangle) S^n(t) dt.
\end{aligned}$$

Since $P(x) = x$, we have that $DP_x(x') = x'$. So by integration by part we get (2). Since, for $S(\cdot) \neq \text{const.}$,

$$(|x'(t)|^2 + S^{-2}(t)|Ax(t)|^2) S^n(t)$$

is not conserved, (2) is a non-autonomous system. \square

Define

$$\Lambda^1 = \Lambda^1(N) = W^{1,2}(S^1, N).$$

It is well-known that $\Lambda^1(N)$ is a Hilbert manifold [P1]. Since N is compact, there exist constants $c_i > 0$ ($i=1, 2, 3$) such that

$$(3) \quad c_1 D(y) - c_3 \leq F(y) \leq c_2 D(y) - c_3,$$

here $D(y) := |y|_1^2 = \int_{S^1} |y'|^2$ for every $y \in \Lambda^1$. We will also need the following inequality

$$(4) \quad |y|_\infty \leq |y(0)| + c_4 |y|_1,$$

for every $y \in \Lambda^1$ and the Sobolev imbedding $W^{1,2}(S^1, R^k) \rightarrow C^0(S^1, R^k)$ is compact.

LEMMA 3. — i) $F(\cdot)$ satisfies Palais and Smale condition C ;

ii) For every $c > 0$, there exists an integer $\bar{n} = \bar{n}(c)$ such that

$$H^n(I_c) = 0$$

for $n > \bar{n}$, where $I_c = D^{-1}(-\infty, c]$.

Proof. — i) Suppose $\{x_m\} \subset \Lambda^1$ is a sequence such that

$$(5) \quad F(x_m) \rightarrow c$$

and

$$(6) \quad dF(x_m) \rightarrow 0, \quad \text{in } H^{-1}.$$

Since N is compact, we may assume that $x_m(0) \rightarrow p$. By inequality (3), we get that, there exist a constant $C > 0$ such that

$$(7) \quad D(x_m) \leq C.$$

So we may assume that $x_m \rightarrow x$ in $C^0(S^1, N)$. Now,

$$\langle dF(x_m), \eta \rangle = 2 \int_{S^1} (\langle x', \eta' \rangle + \langle D^2 P(x', x'), \eta \rangle - S^{-2}(t) \langle A^2 x, \eta \rangle) S^n(t).$$

Take $\eta = x_m - x_n$ and $x = x_m, x_n$ in (6), we get by (7) and (4) that

$$\begin{aligned} o(1) &= 2^{-1} \langle dF(x_m) - dF(x_n), x_m - x_n \rangle \\ &\geq c_1 D(x_m - x_n) - 2C |x_m - x_n|_\infty \\ &\quad - \int_{S^1} S^{n-2}(t) \langle A^2(x_m - x_n), x_m - x_n \rangle \\ &\geq c_1 D(x_m - x_n) - 2C_5 o(1). \end{aligned}$$

Here we implicitly used boundness of the positive function $S(t)$. Hence, $D(x_m - x_n) = o(1)$.

ii) This is borrowed from Milnor's book (see theorem 16.2 in [M]). Since I_c is a strong deformation retract of a finite dimensional manifold, whose dimension n depends on c , then, we get the conclusion if we let $\bar{n}(c) = n$. \square

Now, let us recall a result of M. Vigue-Poirrier and D. Sullivan [V-PS] about the topology of Λ^1 .

PROPOSITION 4. — *If N is compact and simply connected, then there exists an infinite set of positive integers $\mathbb{M} \subset \mathbb{N}$ such that*

$$H^q(\Lambda^1) \neq 0$$

for $q \in \mathbb{M}$.

3. Final argument.

Consider a non-trivial $\alpha \in H^*(\Lambda^1)$ and set

$$(8) \quad \bar{\alpha} = \{B \subset \Lambda^1; i_B^*(\alpha) \neq 0\},$$

where

$$i_B^*: H^*(\Lambda^1) \rightarrow H^*(B)$$

is the homomorphism induced by the inclusion

$$i_B: B \rightarrow \Lambda^1.$$

Remark 5. — $\bar{\alpha}$ defined in (8) is non-empty and contains the compact support of a k -chain $a \in \alpha$, $k = \deg \alpha$, which is not homologous to constant by the nontrivial property of α .

LEMMA 6. — Let $\alpha \in H^*(\Lambda^1)$, $\alpha \neq 0$ and define

$$(9) \quad c_\alpha = \inf_{B \in \bar{\alpha}} \sup F(B).$$

Then, c_α is a critical value of F on Λ^1 ; moreover, if we assume that $H^q(\Lambda^1) \neq 0$ for infinitely many q , there exists a sequence $\{c_\alpha\}$ of critical values of F defined as in (9) which satisfies that

$$(9') \quad c_\alpha \rightarrow +\infty, \quad \text{as } \deg \alpha \rightarrow +\infty.$$

Proof. — By our Remark 5 we have

$$c_\alpha < +\infty.$$

Suppose some c_α is not a critical value of F , then by lemma 3 i) and a well-known deformation lemma in page 125 of R. S. Palais [P1], we know that there exists a positive number ε and a homeomorphism η on Λ^1 such that

$$(10) \quad \eta(F_{c_\alpha+\varepsilon}^{-1}) \subset F_{c_\alpha-\varepsilon}^{-1}.$$

Since

$$\eta^*: H^q(\eta(\Lambda^1)) \rightarrow H^q(\Lambda^1)$$

is an isomorphism, we have that

$$i_{\eta(B)}^*(\alpha) = (\eta^*)^{-1} \cdot i_B(\alpha) \neq 0$$

for all $B \in \bar{\alpha}$. Hence η leaves $\bar{\alpha}$ invariant. But, by the definition of c_α , there exists $B \in \bar{\alpha}$ such that

$$\sup F(B) < c_\alpha + \varepsilon.$$

So by (10) and $\eta(B) \in \bar{\alpha}$ we have

$$\sup F(\eta(B)) < c_\alpha - \varepsilon.$$

It is absurd.

To get (9'), we take $k \in \mathbb{N}$. By lemma 3 ii), there exists $\bar{n} = \bar{n}(k) \in \mathbb{N}$ such that $H^q(I_k) = 0$ for $q > \bar{n}$. By our assumption on $H^*(\Lambda^1)$ we may take $q_k > \bar{n}$ with $H^{q_k}(\Lambda^1) \neq 0$ and consider $\alpha \in H^{q_k}(\Lambda^1)$, $\alpha \neq 0$. Denote

$$I^k = \{x \in \Lambda^1; D(x) > k\},$$

we claim that

$$(11) \quad \forall B \in \bar{\alpha}, B \cap I^k \neq \emptyset.$$

Suppose it is not true, then, there exists $B \in \bar{\alpha}$ such that

$$B \subset \Lambda^1 \setminus I^k := I_k,$$

then

$$(12) \quad H^{q_k}(\Lambda^1) \xrightarrow{i_2^*} H^{q_k}(I_k) \xrightarrow{i_1^*} H^{q_k}(B),$$

where i_2^* , i_1^* are the homomorphisms induced by the inclusion maps

$$i_2: I_k \rightarrow \Lambda^1, \quad i_1: B \rightarrow I_k.$$

Then, by $B \in \bar{\alpha}$ we have that

$$(13) \quad i_1^* \cdot i_2^*(\alpha) = i_B^*(\alpha) \neq 0.$$

From (12) and (13) we obtain that $H^{q_k}(I_k) \neq 0$, a contradiction to our assumption on q_k . So (11) is true.

By (11) and our choices of c_α we have that

$$c_\alpha \geq c_1 k - C$$

which implies our conclusion. \square

Proof of Theorem. — 1) If N is simply-connected, then the result follows from Proposition 4 and Lemma 6.

2) If $\pi_1(N) \neq 0$ and finite. Then the universal covering (\tilde{N}, Π) is compact. By 1) we have infinitely many critical points $\tilde{x}_n: S^1 \rightarrow \tilde{N}$ of F such that

$$F(\tilde{x}_n) \rightarrow +\infty, \quad \text{as } n \rightarrow \infty.$$

Therefore, set $x_n = \Pi(\tilde{x}_n)$, we obtain the existence of infinitely many critical points of F , and infinitely distinct harmonic maps of periodic 2π in t from M to N by Lemma 1.

3) If $\pi_1(N) = \infty$. We may get a minimizer of F in each homotopy class by the Palais-Smale condition in lemma 3 i). \square

Remark 7. — (1) Suppose $S(t)$ is not periodic in t . Take $I = [0, 1]$, $x(0)$ and $x(1)$ two point in N , we can prove as in our theorem that there are infinitely many geometrical distinct critical points of F . It is interesting to consider the behavior of the orbit of some critical point of F just like that of the geodesic in N .

(2) It is an open question to obtain our theorem when $S(t) = 1 - \cos(t)$. In this case, the Lorentz manifold M is called Friedman-Robertson-Walker space-time in general relativity.

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